Braiding Voice Leadings

Representing Voice Leadings and Chord Progressions as Braids

Master Thesis

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Abstract

In this thesis, we investigate a meaningful way to represent voice leadings as braids in the braid group B_n. This rather unconventional approach could allow for new ways to characterise music pieces. The thesis starts by considering two basic braid representations for voice leadings and chord progressions. With these two representations, there is the primary issue of lack of distinction between over- and undercrossings when forming braids. This lack of distinction is arguably necessary to acces all the potential braids in B_n.

The thesis mostly considers voice leadings as bijection $p : X \to Y$ between chords X and Y. Without distinction, the voice leading structure collapses to the symmetric group $S(X)$ or $S(Y)$. In fact, the set of voice leading maps p will be a torsor of the symmetric group $S(X)$ or $S(Y)$. This means that it is equivalent without origin, i.e., it will always be dependent on a chosen base progression p_0. If one considers X and Y to be unordered, there are many isomorphisms to S_n also based on an initially chosen order.

With braids having a natural group homomorphism to S_n, it is natural to consider voice leadings as elements of S_n and consider lifting these to B_n. In order to resolve the issue of distinction between crossings, we introduce ordered changes for a voice leading. Musically, this can be interpreted as arpeggios where changes occur in sequence rather than all at once. By allowing to vary the voice leadings and the order of said voices itself, we introduce extra complexity, such that we get a torsor of the group $S(X) \times S_n$, respectively $S(Y) \times S_n$. With this in mind, we find a unique map that embeds the voice leadings, as elements of $S_n \times S_n$, into B_n. However, we will find that this map is not injective.

We also consider what type of braids the latter map can induce, i.e., braids with a valid order displayed in the strands. We will find some limits on knot invariants regarding these valid braids, concluding that it might be a rather small set of knots that could come from closing such valid braids. We even find an edge case of braids displaying a total order in a minimal amount of crossings, where closing all such braids leads to the unknot, the trivial knot.

Furthermore, we notice that some crossings give redundant information regarding the order of strands, i.e., the order of voices. This redundancy is troublesome since it non-trivially changes the braid whilst displaying no more extra information. If we consider only those braids with no redundancy, we eventually end up again with permutation braids, that can be lifted from S_n to B_n.

We end with a digression on more general, abstract notions of braids, partial singular braids, to allow for more flexible definitions to meaningfully embed musical aspects.

Preface

Starting my master, I had no clear preference in any of the specific mathematics fields. The Technical University of Eindhoven (TU/E) offers many different courses, and I have taken quite a broad selection of them, resulting in an equally broad choice for thesis topics. Luckily, my passion for music has been a steady thread through my academic career. As a guitarist, I have started playing in multiple bands, broaden my musical taste, and even joined the activity committee of Studentproof Jazz (the local student jazz association), all while pursuing my degree in mathematics. Thankfully to some suggestions, I started to look into mathematics for music and found a lot. The connections that can be made with music are almost as broad as mathematics itself. I have read a lot of very interesting papers from the Mathematics and Computation in Music conferences [22]. My interest was guided by two papers discussing ways to compare genres and defining metrics on chords. Reading more into the analysis and algebra of those metrics, I stumbled upon Prof. Dmitri Tymoczko, a mathematician who wrote a lot on the geometry of music. The reading was very intimidating, as it was very abstract and complicated, but I managed to make out some things. I can refer the interested reader to his book *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*. Reviewing more literature, I stumbled upon the doctorate thesis of dr. Mattia Bergomi. This is where I first encountered knot theory. And the connection he made with voice leadings intrigued me very much, even though I did not know anything on the topic. I was not going to ignore this, and started looking to knot and braiding theory. With some helpful recorded lectures from Andrews University (see *Math at Andrews University* on YouTube), I picked up the basics of knot and braiding theory. Now more able to read into the relevant articles, I reread the relevant chapters of dr. Bergomi's thesis. I started experimenting with what I knew, and before I knew it had encountered some issues with this idea. Documenting makes for a good way to start research, and I started looking into what those issues where and why they occurred. In retrospect, it is clear that braiding is not a very optimal way to consider voice leadings, but since Prof. Tymoczko has already provided an extensive bibliography on the algebraic structures, I thought it might be interesting to make some braiding applications. And it was very interesting indeed. The rest is written out in this thesis. There is plenty more that can be said on the braids in this context, but I had to tie an end to this knot. This makes my thesis the work that it is, and I hope you will find this as interesting as I did.

I would like to thank my supervisors, Lisa Kusch and Annika Bach, for their guidance throughout this project and for taking the time to continuously review my work. Also, I would like to thank Alberto Ravagnani and prof. Hans Cuypers for meeting me to discuss some early points on this idea.

Sincerely,
Tuur Willio

Contents

List of Symbols

Table 0.1: List of symbols

Symbol	Description
ν	Frequency, Hertz (Hz)
$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers, starting on one
$\mathbb{Z}$	Whole numbers
$(G, *)$, $(H, *)$	group or monoid structure
S_n	Symmetry group of n elements
π	Permutation, element of S_n
B_n	Braiding group of n strands
$\sigma_i^{\pm 1}$	Braiding group generator, a crossing
β	Braid, element of B_n
$\mathbb{S}_1$	Circle piece $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
K	A knot, a simple closed curve, embedding of $\mathbb{S}_1$ in $\mathbb{R}^3$
χ	Euler characteristics
g	Genus of a connected orientable surface
m	Number of components in a braid or knot
$\mathcal{I}_n$	Partial permutation monoid of n elements
$\mathcal{PSB}_n$	Partial singular braid monoid of n strands
$\mathcal{SB}_n$	Singular braid monoid of n strands
$\mathcal{IB}_n$	Partial braid monoid, or the inverse braid monoid of n strands
$\mathbb{T}^n$	n-dimensional torus, the space of pitch classes
$\mathbb{A}_n$	Chord space, pitch space with permutation equivalence
$\mathcal{P}(X, Y)$	Set of chord progressions from chord X to Y
SB_n	Set of positive permutation braids
s_i	A specific strand i in the braid, independent of the permutation
$o(s_i)$	The order of that specific strand in the braid, when considering ordered strands
$\mathcal{P}_o(X, Y)$	Set of chord progressions from chord X to Y with ordered voice leadings
$\mathcal{P}^*(X, Y)$	Set of chord progressions from ordered chord X to ordered chord Y
$\mathcal{P}_o^*(X, Y)$	Set of chord progressions from ordered chord X to ordered chord Y with ordered voice leadings

*Note that deviation from these definitions is possible, but rare, i.e., symbols might be repurposed depending on the context. The context will be clear on the use of a symbol.

1 | Introduction

In this thesis, we look into an interesting way to characterise musical voice leadings and chord progressions. We do this by looking into braiding theory as a candidate. Braiding theory is an algebraic topological field in mathematics that studies knots in form of braids. The braiding group is the set of equivalence classes of loops under homotopy of points in a two-dimensional Euclidean space. This means that we can represent braids as two-dimensional points moving along a third dimension, time. This explains why we can visualise braids as three-dimensional objects.

Musical voice leadings, on the other hand, also evolve over time. We can thus investigate if we have a good representation in braids for voice leadings.

1.1 | Literature Overview

The idea of using braids in this context is not new and has been suggested before by Dr. Mattia Bergomi in his PhD thesis [6] and a preprint [5]. However, it is only used quite lightly. In this thesis, we will thus look more into this idea of using braids as a way to characterise chord progressions and voice leadings.

As mentioned in the preface, there is more literature on the connection between music and mathematics. The paper [16] makes a collection of numerous metrics, defined to evaluate distances between musical chords. These metrics are specifically constructed to compare chords of different sizes. Examples of such metrics, are those provided in [12] that are based on projections and extensions of said chords. It is important to note that most of these metrics in [16] are pseudo-metrics and each have some drawback to using them.

Also, as previously mentioned, the general geometrics of musical spaces is already discussed extensively by Dmitri Tymoczko [13, 7]. Musical spaces are those of equivalence classes based on octave equivalence, permutation equivalence, transpose equivalence, inverse equivalence, and even cardinality equivalence, with the latter already hinting at extension or projection chord.

Obviously, we require some literature on general knot theory and braiding theory. These were all provided by following books and articles [18, 20, 8, 19, 2, 4, 15]. Some more specific theory on permutation braids was acquired using [1, 21].

Images in this thesis from some knots and all Seifert surfaces were generated using the software SeifertView from the Technical University of Eindhoven [23]. The paper [23] also provided some handy formulas for the knot genus based on the initial surface and its closed variant.

In the digression, in Chapter 6, we talk about more general forms of braids, i.e., partial singular braids. The relevant theory on these braids was provided in [11, 10, 9, 24]

1.2 | Motivation

The idea of representing voice leadings and chord progressions as braids might give new perspective on how to characterise these musical elements. This characterisation could aid in analysing song structures, and quantifying stylistic choices.

We consider voice leadings and chord progressions. Both are similar to each other but not quite. Voice leadings are a set of voices, e.g., different instruments, that each have their own melody line. This way, the voices each play their own melody line simultaneous but independently. A chord progression is similar in the way that we do not distinguish between voices. With voice leadings, we can clearly assign melodies and notes to certain voices. For chord progression, we just consider a set of notes, chords, that all simultaneously change into another chord.

Clearly, the concept of representing voice leadings and chord progressions as braids is significantly different from what is known and done in the above-mentioned literature regarding musical representations. Optimally, one could create new pseudo-metrics based on knot and braid invariants to characterise the complexity of certain voice leadings and chord progressions in order to distinguish them.

1.3 | Chapter Overview

We start in Chapter 2 by rigorously introducing the theory on knots, braids, some group theory and the relevant topology in order to understand the concept of knots and braids. We will not, however, be actively using all this theory, but it is given as a way to be self-contained and provide the reader with an appropriate context.

In Chapter 3, we look into some naive representations of voice leadings as suggested in [6] and a way to braid chord progressions as in [5]. More specifically, in Section 3.1, we introduce a way to braid voice leadings based on voice crossings, and in Section 3.2 we look into the braid representation of article [5] for chord progressions. We look into some weaknesses of these representations based on the musical context.

From this chapter onward, we will look into why such weaknesses occur and whether we can do something about it. We will define generally what it means to be a voice leading by constructing a set of valid progressions that embody said voice leadings.

In Chapter 4, we thus consider a more general definition of voice leadings and prove its many non-canonical equivalences with the symmetric group. In Section 4.1, we then look into how this group can be embodied in the braiding group, and what the braids look like as the result of many non-canonical equivalences.

In Chapter 5, we make up for the lack of structure that was introduced earlier in Chapter 4 by adding an order on the voice leadings, and thus an order on the braid strands. This also suggests a fix for the main issue with the representation in Section 3.1. Musically, an order on the voice leadings is like an arpeggio where voices change in order, not all at the same time.

With this idea of adding order, in Section 5.1, we look into what kind of braids adhere to such an ordering, independent of the musical context, as a way to capture the limits of such a representation. We will come to the conclusion that many such braids can adhere to an order structure and that it is rather hard to explicitly consider all such braids. We can, however, get some grasp on the braids with ordered strands that are so-called non-redundant. This will be defined at the appropriate moment, in Definition 5.27, near the end of Section 5.1. It involves considering braids that do not allow for strands to cross more than once, since a single crossing is required to define the relation on the two affected strands. Another crossing is either redundant in the sense that it provides information that is already know, or induces the opposite information and becomes a contradiction with respect to (w.r.t) order relation.

Also, we will consider an edge case that clearly embodies this idea of 'limited representation' in the braiding group. Furthermore, we look into some bounds on some previously defined knot and braid invariants, for braids with order on the strands.

As we did in Chapter 4, in Section 5.2, we also look in a more general way to braids with ordered strands. We find some additional structure, i.e., an equivalence with the Cartesian product of two symmetric groups. We still have a lot of non-natural isomorphisms, i.e., we have the same issues as we had in Chapter 4. A lack of 'origin' results in all elements in the set being a valid origin, each inducing its own equivalence with said symmetric groups.

In Section 5.3, we add some more structure by using ordered voice leadings of ordered chords, where the order of the notes matter. The order of the notes in a chord could matter, musically, by differencing between low and high pitches, as to allow for distinct inversions of a chord. A chord inversion is when the order of the notes are changed, e.g., one could have a note in the base that was initially not the root note of a given chord. Chord inversions allow for more spice in a song. This added structure will make for a canonical equivalence but is still dependent on choosing a base ordering on the chords or a base ordered voice leading.

In Section 5.4, we again consider a map from the Cartesian product of the symmetric group to the braiding group. This allows us to refine the initial map we had in Section 4.1. We also add more structure by embedding the chord distance, as a result of the natural metric on the chord space, into the braid. This way, more information is present in the braiding representation.

In Chapter 6, we digress from the main results and ideas to wonder about using partial singular braids in this musical context. We consider chord progressions between chords of different sizes and more representation that use partial singular braids. This chapter serves as a collection of what is already suggested in the PhD of Bergomi[6], and some alternative more general approaches. Since this is a digression it is quite limited in depth.

We end this thesis with our conclusions in Chapter 7.

2 | Preliminaries

We will introduce the relevant ideas on voice leadings, group theory, topology, partial orders, and braiding and knot theory. Next to what is provided below, there is obviously more theory available. For more information on these topics, one may refer to the following papers and books [6, 7, 15, 18, 20, 8, 19] for information on the formal description of music, and knot and braiding theory, respectively.

2.1 | Pitches and Voiceleadings

2.1.1 | Quantisation of pitches

To characterise musical notes and voice leadings we must first quantise the actual notes. Modern music utilises a 12 note system in equal temperament. This means that for an octave, which is the interval between a note and a higher or lower equivalent of this note, there are 12 notes in this interval. The term equal temperament then denotes that the 12 notes are spread equally (on a logarithmic scale) over the octave. The 12 distinct notes[1] can be denoted as

$$C,\ C^\sharp,\ D,\ D^\sharp,\ E,\ F,\ F^\sharp,\ G,\ G^\sharp,\ A,\ A^\sharp,\ B. \tag{2.1}$$

This is one way to notate these notes. One could write $D^\flat$ instead of $C^\sharp$ since they have the same pitches. This principle is called enharmonic equivalence, where different notated notes have the same pitch. Notation differs for ease of reading or depend on what musical scale is used. We will generally not make a distinction here. To refer to a specific note, in a specific octave interval, we can add a subscript to denote the relevant octave, e.g. C_1 is the C note in the first octave, A_4 is the A note in the fourth octave.

If we start at a base note, that we signify with value 1, its octave is 2 and all the 12 notes in between are $2^{i/12}$, for $i \in [0, 12)$. This shows that every next note is a $2^{1/12}$ factor higher. This distance is often called a half step or half note.

The pitches of notes are physically determined by their frequency, and it is this frequency that thus increases exponentially with the factors $2^{1/12}$. Our hearing range as humans goes from 20 to 20000 Hertz (Hz). This range is then divided in octaves such that octaves differ in frequency by a factor of 2, e.g. 20Hz to 40Hz is an octave. The modern tuning standard takes A_4, the *La* note in the fourth octave, as the note with a frequency of 440 Hz. Since the increase in frequency is exponential we can define a logarithmic function that is thus linear and additive for pitches. The function is defined from $(0, +\infty)$ to $\mathbb{R}$ with:

$$p(\nu) = 49 + 12\log_2\left(\frac{\nu}{440}\right) \qquad \forall \nu \in (0, \infty). \tag{2.2}$$

The factor of 12 is clear to denote that the octave contains 12 notes from $12\log_2(1)$ to $12\log_2(2)$. If we have pitches with frequencies of the form $2^{k/12}440$, this will result in $p(2^{k/12}440) = 49 + k$. The constant of 49 indicates that the 49th key on the piano is the A_4 note. This makes that we now have a system to identify (piano)keys for certain pitches. We could make a range of p functions for different instruments that have ordered notes, ordering in the sense of pitch. For example, for a guitar we require 6 different functions to denote the six different strings, e.g. the lowest E string we have a function $p_g(\nu) = 12\log_2(\frac{\nu}{82})$.

[1] Modern music is not limited to just 12 distinctive notes. Bending notes and slidings of pitches create a continuous change, allowing for pitches in between the 12 notes.

The idea is that we cannot distinguish the absolute frequencies, but rather their ratios and thus the logarithm makes more sense to consider. Our perception of changes in pitch is now more linear with the changes in $p(\nu)$.

From now on, we will generally use the value of $p(\nu) \in \mathbb{R}$ as pitches. Since the term of 49 in p was piano specific, we will not care about the actual value of $p(\nu)$, but rather just the logarithm, e.g. 0 is a designated note, 12 is one octave higher, and 24 two octaves higher.

2.1.2 | Voiceleadings

Now that we have a way to represent the notes, we can define voice leadings. Mathematically, a voice leading can be defined as follows:

Definition 2.1. *A **voice leading** is a collection of melodies that can be viewed as independent parts that occur simultaneously. Mathematically, a voice leading with $k \in \mathbb{N}$ voices can be defined as follows*

$$\left\{v_i : \{0, 1, \dots, T\} \to (0, \infty) \,|\, 0 < i \leq k,\, i \in \mathbb{N}\right\} \tag{2.3}$$

Here T is the discrete time duration of the piece and $v_i(t)$ denotes the pitch played by voice i at time t. Here, rhythm is discretised in a way that is representative, e.g., if t are steps of 16th notes, an 8th note occurs twice. When considering certain changes we can think of a voice leading as mapping one note to another, i.e., for k voices we can make an amount of T progression bijections, such that for $t \in \{1, \dots, T\}$:

$$p_t : X_{t-1} \to X_t : x_{t-1} \mapsto x_t, \tag{2.4}$$

where X_t denotes the set of notes occurring at time t such that there is a voice going from note $x_{t-1} \in X_{t-1}$ to $x_t = p_t(x_{t-1})$. We have $|X_t| = k$.

Here we consider that if a voice is silent, no note is played, the previous note is extended until the occurrence of the next note. An example of a voice leading is shown in Figure 2.1.

Example 2.2. In Figure 2.2, there is a one bar piece for two voices. Describing this mathematically with Definition 2.1, taking zero as C_3, we get the following:

$$v_1 = (\overbrace{28, 28, 28, 28}^{\text{quarter note } E_5}, \overbrace{31, 31, 31, 31}^{\text{quarter note } G_5}, \overbrace{\underbrace{31, \dots, 31}_{6 \text{ times}}}^{\text{extension until next note}}, \overbrace{24, 24}^{\text{eight note } C_5}), \tag{2.5}$$

$$v_2 = (12, 12, 19, 19, 16, 16, 24, 24, 5, 12, 4, 11, 23, 23, 12, 12), \tag{2.6}$$

with $T = 15$ because we have 16 16th notes in one bar and the first voice, v_1, being the top in Figure 2.2. Clearly, all notes are discretised to 16th notes, e.g., the quarter note E_5 in the beginning for voice one occurs four times because there are four 16th notes in a quarter note. Also clearly, the last note gets extended until the next occurrence, as happened for G_5 for the first voice.

The bijection-presentation of the voices give for example the following map for $t = 4$:

$$p_4 : \{28, 19\} \to \{31, 16\}, \tag{2.7}$$

that maps 28 to 31, and 19 to 16, according to the discretisation of the voice leading.

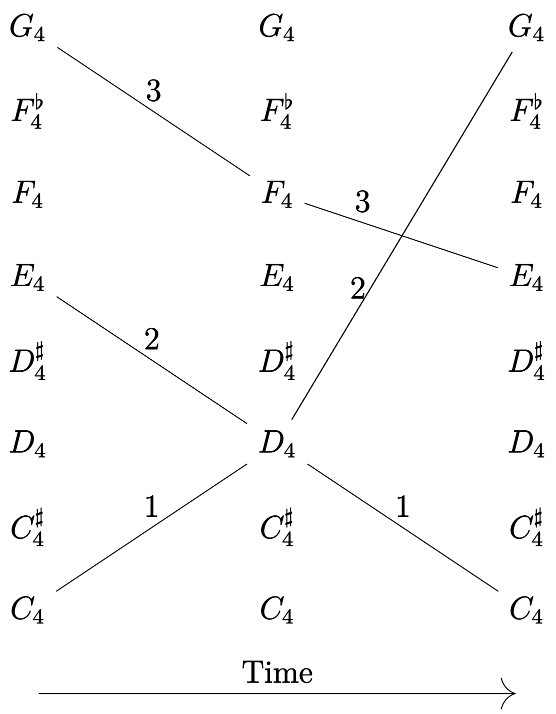


Figure 2.1: An example of a voice leading with three voices going $C_4 \mapsto D_4 \mapsto G_4$, $E_4 \mapsto D_4 \mapsto C_4$, and $G_4 \mapsto F_4 \mapsto E_4$ respectively.

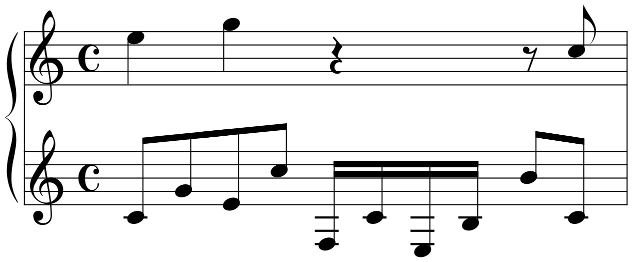


Figure 2.2: Short music piece with two voices

With a discrete sequence of notes, one can make a scheme as shown in Figure 2.1 where voice changes are indicated by lines.

There are lots of things that can be said about the practice of voice leadings and motions of the voices[6], but for now we leave this open. However, we will consider voice crossings. We shall define this as follows:

Definition 2.3. *A **voice crossing** is a crossing in pitch between two voices. That is, for two voices i and j, there exists $t_1, t_2 \in \{0, 1, \ldots, T\}$ with $t_1 < t_2$ such that:*

$$v_i(t_1) > v_j(t_1) \qquad \text{and} \qquad v_i(t_2) < v_j(t_2)$$

Example 2.4. An example of a voice crossing is shown in Figure 2.1. The voices are respectively $(0, 2, 7)$, $(4, 2, 0)$, and $(7, 5, 4)$ for C_4 as zero. Clearly voices 2 and 3 cross in between $t_1 = 1$ and $t_2 = 2$.

2.2 | Group Theory

In order to discuss knot theory and the relevancy of certain presentation, we require some group theory. We review some of these concepts now.

Definition 2.5. *A* ***group*** $(G, *)$ *is a set* G *and an operation* $*$ *such that* G *is closed under* $*$*, i.e.,* $\forall g, h \in G:\ g * h \in G$*, and satisfying following properties:*

- *(Associativity)* $\forall g_1, g_2, g_3 \in G:\ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$,
- *(neutral element)* $\exists\, 1 \in G,\ \forall\, g \in G : 1 * g = g * 1 = g$,
- *(inverse element)* $\forall\, g \in G,\ \exists\, g^{-1} \in G:\ g * g^{-1} = g^{-1}g = 1 \in G$.

When there is no confusion the group operator $*$ *can be abbreviated to the multiplicative notation, i.e.,* $g * h$ *becomes* gh*, for all* $g, h \in G$*. Equivalently, the operator can also be abbreviated to additive notation, e.g.,* $g + h$ *for* $g, h \in G$*, with neutral element zero, and inverse elements* $(-g)$*.*
If the operation $*$ *is commutative, i.e.,* $gh = hg$ *for* $g, h \in G$*,* $(G, *)$ *is a commutative or Abelian group.*

Remark 2.6.

1. Note that from the neutral element condition it is clear that the existing neutral element is unique. For e_1, e_2, two neutral elements, we have $e_1e_2 = e_1 = e_2$.
2. Multiplicative and additive notations for specific groups are a convention, but are both representing the same idea.

Example 2.7. An example of a group are the modulo equivalence classes of whole numbers. For example, for $n \in \mathbb{N}$, $\mathbb{Z}_n = \{z \bmod n \mid z \in \mathbb{Z}\} = \{[0], \ldots, [n-1]\}$ is a group with operation $a * b = (a + b) \bmod n$ and neutral element 0. The elements of $\mathbb{Z}_n$ can also be written as $\{\overline{0}, \ldots, \overline{n-1}\}$.

Let us define the following additional operator on $\mathbb{Z}_n$, with $z \in \mathbb{Z}_n$ and $k \in \mathbb{Z}^+$:

$$k \cdot z := \underbrace{z + z + \cdots + z}_{k \text{ times}}. \tag{2.8a}$$

If $k \in \mathbb{Z}^-$ we define:

$$k \cdot z := \underbrace{(-z) + (-z) + \cdots + (-z)}_{k \text{ times}}. \tag{2.8b}$$

Note that this is not per se a new operator but rather a shorter notation for longer summations. With this in mind, it is now clear that any element in $\mathbb{Z}_n$ can we be written as a $k \cdot 1$ for a specific $k \in \mathbb{Z}$. For example, $\overline{3}$ can we be written as $3 \cdot \overline{1}$. The neutral element $\overline{0}$ can be written as $0 \cdot 1$. The neutral element of $\mathbb{Z}_n$ becomes $1 \cdot 1$.

If all group elements can be generated such as in the above example, we talk about group generators.

Definition 2.8. *A group* $(G, *)$ *can be generated by a set of elements* $M = \{g_1, \ldots, g_m\} \subset G$*, called* ***group generators****, if for all* $g \in G$ *there exists integers* $k_1, \ldots, k_m \in \mathbb{Z}$ *such that*

$$g = g_1^{k_1} g_2^{k_2} \ldots g_m^{k_m}, \tag{2.9}$$

where, for $k \in \mathbb{Z}^+$*, we have*

$$g_i^k = \underbrace{g_i * \cdots * g_i}_{k \text{ times}}, \tag{2.10a}$$

and for $k \in \mathbb{Z}^-$, we have

$$g_i^k = \underbrace{g_i^{-1} * \cdots * g_i^{-1}}_{k \text{ times}}. \tag{2.10b}$$

For $k = 0$, we define $g^k = 1$, with 1 the neutral element.

The operations here are equivalent to what was already defined in (2.8).

We notate $G = \langle M \rangle$ or $G = \langle g_1, g_2, \ldots, g_m \rangle$.

*If a group G has one generator, i.e., $G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$, it is called a **cyclic group**. The generator cycles through all elements of the group.*

*If in general M is finite, we say that G is **finitely generated**.*

Remark 2.9. Note that a subset M of G only makes sense as generators of group G, if M is a proper subset, i.e., M is strictly smaller than G. Obviously, an element g can be generated by itself, i.e., $g = g^1$.

It is now clear that the modulo group in Example 2.7 is cyclic.
Another important example of a group, one that we will use extensively, is the following:

Definition 2.10. *For $n \in \mathbb{N}$, the **symmetric group** S_n consists of all permutations of n elements. All permutations can then be written down using just n numbers, 1 to n. A permutation is a bijective map π*

$$\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}. \tag{2.11}$$

An element $\pi \in S_n$ permutes a value i to $\pi(i)$. All permutations can be notated as cycles. A cycle of length k, also k-cycle, has the following structure, for an arbitrary $x \in \{1, \ldots, n\}$:

$$(x \ \pi(x) \ \pi^2(x) \ \ldots \ \pi^k(x)), \tag{2.12}$$

with $\pi^{k+1}(x) = x$. Each element is permuted to the next in the cycle.

The neutral element is 1 being the trivial permutation $\pi = Id$. Since the elements are function, the natural operator is a composition of permutations, denoted by $\circ$.

When considering permutations of a set X, the group is notated $S(X)$.

Example 2.11. For a notation example, $(1\ 2\ 4)(5\ 3)$ permutes one to two, two to four, four to one, and three to five, and five to three. A composition example is $(1\ 2) \circ (1\ 2) = Id$ and $(1\ 4\ 2) \circ (1\ 3) = (1\ 3\ 4\ 2)$. Notice that any cycles with disjoint element can be commutated, e.g., $(1\ 2)(3\ 4) = (3\ 4)(1\ 2)$.

We can prove the following for the generators of the symmetric group S_n.

Theorem 2.12. *The symmetric group S_n is generated by elements $(i\ i+1) \in S_n$ with $1 \le i \le n-1$, such that:*

- $(i\ i+1)^2 = Id$ *for all* $1 \le i \le n-1$
- $(i\ i+1)(i+1\ i+2)(i\ i+1) = (i+1\ i+2)(i\ i+1)(i+1\ i+2)$ *for all* $1 \le i \le n-2$
- $(i\ i+1)(j\ j+1) = (j\ j+1)(i\ i+1)$ *for all* $|i-j| > 1$ *with* $1 \le i, j \le n-1$

Proof. First, note that all 2-cycles of the form $(i\ j)$ can be constructed as follows:

$$(i\ j) = (i{+}1\ i)(i{+}1\ i{+}2)\dots(j{-}1\ j{-}2)(j{-}1\ j)(j{-}1\ j{-}2)\dots(i{+}1\ i{+}2)(i{+}1\ i). \quad (2.13)$$

Also note that the cycle $(i\ i+1)$ is the same as the cycle $(i+1\ i)$. Thus, if we can show that the 2-cycles $(j\ k)$ generate all permutations, we have proven what has been asked. Now any cycle $\pi = (x\ \pi(x)\ \pi^2(x)\ \dots\ \pi^k(x))$, with $\pi^{k+1}(x) = x$, can be deconstructed as follows:

$$(x\ \pi(x)\ \pi^2(x)\ \dots\ \pi^k(x)) \overset{(a)}{=} (\pi(x)\ \pi^2(x)\ \dots\ \pi^{k-1}(x)) \circ (x\ \pi^k(x)) \circ (x\ \pi^{k-1}(x)) \quad (2.14)$$

$$\overset{\text{Repetition of }(a)}{=} (\pi^2(x)\ \dots\ \pi^{k-2}(x)) \circ (\pi(x)\ \pi^{k-1}(x)) \circ (\pi(x)\ \pi^{k-2}(x)) \circ (x\ \pi^k(x)) \circ (x\ \pi^{k-1}(x)) \quad (2.15)$$

This can be repeated until all cycles are 2-cycles. For example, $(1\ 2\ 3\ 4) = (2\ 3) \circ (1\ 4) \circ (1\ 3)$ and $(1\ 2\ 3) = (2) \circ (1\ 3) \circ (1\ 2) = (1\ 3) \circ (1\ 2)$.

Hence, all permutations can be written as 2-cycles and thus also cycles of the form $(i\ i+1)$.

As for the relations described, it is clear that any 2-cycle has order two, i.e., $(i\ j)^2 = Id$, since interchanging i for j, and j for i again results in the identity. For the second relation, let us consider the mappings of elements $i,\ i+1,\ i+2$ respectively for both permutations:

- permutation $(i\ i+1)(i+1\ i+2)(i\ i+1)$
  - maps i to $i+1$, to $i+2$,
  - maps $i+1$ to i, to $i+1$,
  - maps $i+2$ to $i+1$, to i,
- permutation $(i+1\ i+2)(i\ i+1)(i+1\ i+2)$
  - maps i to $i+1$, to $i+2$,
  - maps $i+1$ to $i+2$, to $i+1$,
  - maps $i+2$ to $i+1$, to i.

Clearly now, they are the same permutation.

For the last relation, note that with $|i-j| > 1$, we have that $j+1 > j > i+1 > i$. This means that these cycles permute on different numbers, independent of each other, i.e., disjoint, and can therefore be commuted without changing the permutation. □

A more general notion of groups can be defined by monoids. A monoid has the following definition:

Definition 2.13. *A* ***monoid*** $(G, *)$ *is a set* G *and an operation* $*$ *such that* G *is closed under* $*$*, i.e.,* $\forall g, h \in G :\ g * h \in G$*, and satisfying following properties:*

- *(Associativity)* $\forall g_1, g_2, g_3 \in G :\ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$,
- *(neutral element)* $\exists 1 \in G,\ \forall g \in G : 1 * g = g * 1 = g$.

This is essentially the group structure without the need for inverse elements.

Maps can be defined between groups and some maps preserve structure, like a homomorphism.

Definition 2.14. *For groups $(G, *_G)$ and $(H, *_H)$, a **homomorphism** is a map $f : G \to H$ from group G to H such that, for all $g_1, g_2 \in G$, we have:*

$$f(g_1 *_G g_2) = f(g_1) *_H f(g_2). \tag{2.16}$$

If such a map f exists, we say that G is homomorphic to H.

If f is a bijective homomorphism, it is an isomorphism. We notate $G \cong H$ and say that G is equivalent to H. If $G = H$ and thus f maps to itself, f is an endomorphism. If f is both an endomorphism and an isomorphism, it is an automorphism.

A group can also act on other sets. This is called a group action.

Definition 2.15. *Let X be a set and $(G, *)$ a group. A function $f : G \times X \to X : (g, x) \mapsto g \cdot x$ is an action from G on X if we have:*

- *$1 \cdot x = x$ for all $x \in X$, and 1 the identity element of $(G, *)$,*
- *$g \cdot (h \cdot x) = (g * h) \cdot x$ for all $g, h \in G$ and $x \in X$.*

We can define f similarly by $f' : X \times G \to X$. We distinguish between the two by notating left action and right action respectively.

*An action is **free**, if $g \cdot x = x$ implies that g is the identity element of $(G, *)$ for any $x \in X$.*

Remark 2.16. Note that the product $\cdot$ does not per se have to be the same as the group product $*$. As long as the map is well-defined there is no issue.

Definition 2.17. *An **orbit** Gx, for a specific element $x \in X$, is a subset of X defined by an action of G on X as follows:*

$$Gx = \{g \cdot x \mid g \in G\}. \tag{2.17}$$

*A group action is **transitive** if there is an element $x \in X$ such that $Gx = X$.*

Definition 2.18. *A set X is called a **principal homogeneous space** or G**-torsor** if a group $(G, *)$ acts both free and transitive on X.*

2.3 | Topology

To understand the equivalence of knots and braids, we also require some topology. In topology, spaces are called equivalent if they can be continuously deformed into one another. This is done by a homeomorphism (note the difference with homomorphism!) between topological spaces. Topological spaces are those that come with a topology, i.e., a way of defining open and closed sets, and in this way a notion of continuity.

Definition 2.19. *A **homeomorphism** is a map $f : X \to Y$ from topological spaces X to Y such that:*

- *f is continuous,*
- *f is bijective,*

- *and f has a continuous inverse function f^{-1}.*

If such a map f exists, we say that X is homeomorphic to Y and notate $X \cong Y$.

If two spaces are homeomorphic, they are equivalent in the sense that they can be continuously deformed into one another.

Remark 2.20. 1. Note that a set X with two different topologies gives two different topological spaces, and thus can be homeomorphic or not depending on the topology.

2. Observe that, if well-defined, there is no confusion between $X \cong Y$ (homeomorphism) and $G \cong H$ (group isomorphism) since one is between topological spaces and one is between groups.

Sometimes a space is not one-to-one, but merely a subspace of another space. This can be explained with embeddings.

Definition 2.21. *An* ***embedding*** *from a topological space X to Y, is a continuous injective map $f : X \to Y$ such that $f : X \to f(X)$ is a homeomorphism. If such a map f exists, X embeds into Y and we notate $X \hookrightarrow Y$.*

If X embeds in Y, the space X is practically a subspace of Y.

Mappings from and to topological spaces can also be equivalent, or so-called homotopic.

Definition 2.22. *Two continuous maps f, g mapping from and to topological spaces X, Y respectively are* ***homotopic*** *if there exists a continuous map $H : X \times [0, 1] \to Y$ such that:*

- $H(x, 0) = f(x)$,
- $H(x, 1) = g(x)$

Note that thus for $t \in [0, 1]$ the function $h_t : X \to Y : x \mapsto H(x, t)$ is continuous and on top of that, for a fixed $x \in X$, the functions $h_x : [0, 1] \to Y : t \mapsto H(x, t)$ is continuous. If such a map H exists, we have f homotopic to g and we notate $f \simeq g$.

Remark 2.23. Note that the intermediate maps $h_t(x)$ just need to be continuous and no assumptions are made on injectivity or surjectivity.

Definition 2.24. *Two topological spaces X and Y are* ***homotopic equivalent*** *if there exist maps $f : X \to Y$ and $g : Y \to X$ such that:*

- *$f \circ g$ is homotopic with the identity function in Y , $f \circ g \simeq Id_Y$,*
- *$g \circ f$ is homotopic with the identity functions in X, $g \circ f \simeq Id_X$.*

With such a map f, we have X homotopic equivalent to Y and we notate $X \simeq Y$.

The difference between homeomorphic and homotopic equivalent is that maps need to be bijective between X and Y for a homeomorphism, but not for a homotopy. For a homotopy, just continuity is required. Thus, homeomorphic spaces are also homotopic equivalent.

Example 2.25. The book [19, p114] gives examples of how $\mathbb{R}^n$ is homotopic equivalent to a single point in $\mathbb{R}^n$. All points can be continuously mapped to a single point that makes sense with Definition 2.24, but obviously such a map is not injective, and thus not bijective. This shows that $\mathbb{R}^n$ is homotopic equivalent with a single point, but not homeomorphic.

A homotopy between embeddings is called an isotopy. This has some additional constraints.

Definition 2.26. *A homotopy H between embeddings from X into Y is an **isotopy** if all intermediate maps $h_t(x)$ are also embeddings, that is, also injective and when restricted to the image of $h_t(x)$, with continuous inverse.*

2.4 | Ordered Sets

In Chapter 5, we consider ordered chords and ordered voice leadings. Let us quickly introduce some ordering concept. Let us start with the following definition

Definition 2.27. *A set X is **partially ordered** by $\preceq$ if we have the following properties:*

- *Reflexivity:* $\forall x \in X : x \preceq x$,
- *Transitivity:* $\forall x, y, z \in X : \Big[x \preceq y,\, y \preceq z\Big] \Rightarrow x \preceq z$,
- *Antisymmetry:* $\forall x, y \in X : \Big[x \preceq y,\, y \preceq x\Big] \Rightarrow x = y$.

A partially ordered set can be notated as $(X, \preceq)$. One often refers to this as a poset.

Definition 2.28. *A set X is **strictly partially ordered** by $\prec$ if we have the following properties:*

- *Irreflexivity:* $\forall x \in X : \neg\Big[x \prec x\Big]$,
- *Transitivity:* $\forall x, y, z \in X : \Big[x \prec y,\, y \prec z\Big] \Rightarrow x \prec z$,
- *Asymmetry:* $\forall x, y \in X : \Big[x \prec y\Big] \Rightarrow \neg\Big[y \prec x\Big]$.

A strictly partially ordered set can be notated as $(X, \prec)$.

Note that this does not mean that all elements of X are ordered, but rather that the above properties hold on elements that are ordered. A set where all elements are ordered is a totally ordered set or chain:

Definition 2.29. *A **totally ordered** set or **chain** is partially ordered set $(X, \preceq)$ with the extra condition:*

$$\forall x, y \in X : x \preceq y \lor y \preceq x. \tag{2.18}$$

We can define something similar for a strict total order $(X, \prec)$.

With this definition subsets of partially ordered sets can be chains with a total order on the elements of that particular subset.

Definition 2.30. *An element x in a chain $(X, \preceq)$ is called an **upper bound**, if for all $y \in X$, we have $y \preceq x$.*
*Similarly, we have a **lower bound** x if $x \preceq y$, for all $y \in X$.*
*An element x is a **maximum element**, if $x \preceq y$ implies $y = x$, for all $y \in X$.*
*Similarly, we have a **minimal element** x, if $y \preceq x$ implies $y = x$, for all $y \in X$.*
We can define something similar for a strict chain $(X, \prec)$.

We will work with finite sets, and thus there will always be a minimum and maximal element in a chain. An important result is the following:

Proposition 2.31. *A partially ordered set* $(X, \preceq)$ *has a dual* $(X, \succeq)$ *that is also a partially ordered set. This means that* $x \preceq y$ *becomes* $x \succeq y$*.*

Lower bounds become upper bounds and vice versa, and minimal elements become maximal element and vice versa.
We can do something similar for a strict total order $(X, \prec)$ *and get* $(X, \succ)$*.*

This result is clear for finite sets, which we will consider.

Remark 2.32. In the rest of this thesis, we will introduce a map $o : X \to \{1, \dots, |X|\}$ that denotes the order of elements. This is well-defined as we only consider finite sets X. We do this to avoid confusion with other notation and to allow composition with permutation in order to change the order.

If we have $x \preceq y$, we then write $o(x) \leq o(y)$., e.g., $o(x) = 1$ and $o(y) = 2$. More specifically, we will consider 'ordered' layers, and we will choose as convention that an element x with $o(x) = 1$ is on the first, top layer.

2.5 | Braids and Knots

Finally, we consider actual braiding and knot theory. Since we will use braids more prominently, let us start with that. There is more than one definition for braids, but we will use the following definition by Artin. Hence, this is also often called the Artin braid group.

Definition 2.33. *For* $n \in \mathbb{N}$*, signifying the number of strands in a braid, the* ***Artin*** n***-braid group*** B_n *is group structure given by generators* $\{\sigma_1, \dots, \sigma_{n-1}\}$*, where* σ_i *denotes an overcrossing of the strand at the ith position over the* $(i+1)$*th strand. Similarly,* σ_i^{-1} *is an undercrossing of string* i *with* $i+1$*. The group operator is a concatenation of the braids. The generators have the following relations:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad \text{for all } i \text{ and } j \text{ such that } |i - j| > 1, \tag{2.19a}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \text{for all } 1 \leq i < n - 1. \tag{2.19b}$$

Elements of the group are either denoted by their generators or more generally by β*, i.e.,* $\beta \in B_n$*.*

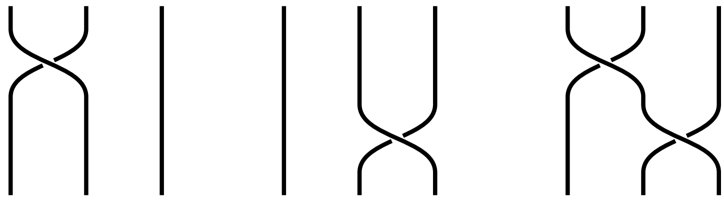
Braid diagrams are read from top to bottom, or left to right. The braiding group is finitely generated.

Example 2.34. The braids $\beta_1 = \sigma_1$ and $\beta_2 = \sigma_2$ are braids in B_3 with product, $\beta_1 * \beta_2 = \sigma_1 \sigma_2$. These braids are shown respectively in Figure 2.3.

Remark 2.35. From now onwards, we will drop the $*$-notation as operator in B_n and just write it as a multiplication, i.e., $\beta_1 \beta_2$.

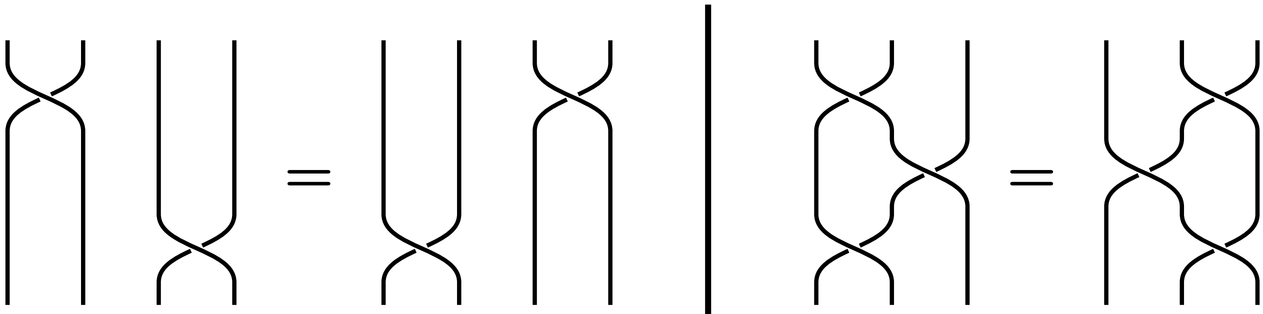
The relation (2.19a) in Definition 2.33 signifies that certain crossings happen separately from each other, on different strands, and can therefore be switched in order, as they do not interact. The second relation (2.19b) is another equivalence. These are both shown in Figure 2.4. If we talk about a braid word, we mean the specific generators that make up a braid. For example, the braid word of $\beta = \sigma_1 \sigma_2 \sigma_1$ is $\sigma_1 \sigma_2 \sigma_1$, and even though it is equivalent with the braid $\sigma_2 \sigma_1 \sigma_2$, we make a distinction in the specific used generators. However, we will only consider braid words up to equivalence, and thus this distinction does not matter.

Let us show that B_n is in fact a group.



β_1 β_2 $\beta_1 * \beta_2$

Figure 2.3: Three braids, $\beta_1 = \sigma_1$, $\beta_2 = \sigma_2$, and $\beta_1 * \beta_2 = \sigma_1\sigma_2$ respectively.



$\sigma_1\sigma_3 = \sigma_3\sigma_1$ $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

Figure 2.4: Artin braid group relations. The braids are read from top to bottom, following the direction indicated by the separating arrow.

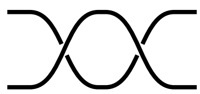
Proposition 2.36. *The Artin braid group for n strands with concatenation operator is a group by Definition 2.5.*

Proof.

- (Associativity)
  Because the operation is braid concatenation, it is clear that for any braids $\beta_1, \beta_2, \beta_3$ in B_n we have
  $$(\beta_1\beta_2)\beta_3 = \beta_1(\beta_2\beta_3). \tag{2.20}$$

- (Neutral element)
  We take as neutral element the trivial braid 1, i.e., no crossings of any kind. Applying this to a braid $\beta \in B_n$ is like extending the strands before or after the crossing. Thus, we can say $1 * \beta = \beta * 1 = \beta$ for any $\beta \in B_n$.

- (Inverse element)
  For any crossing $\sigma_i^{\pm 1}$ there exists the opposite crossing $\sigma_i^{\mp 1}$. Notice, that taking $\sigma_i\sigma_i^{-1}$ or $\sigma_i^{-1}\sigma_i$ has no real crossing. For $\sigma_i\sigma_i^{-1}$, strand i goes over $i+1$, followed by strand i'[2] going under $(i+1)'$, but notice that strand i' is the initial strand $i+1$ and strand $(i+1)'$ is the initial ith strand. This essentially cancels the crossing. To visually understand this, the braid word $\sigma_1\sigma_1^{-1}$ is represented by the following braid:

  (2.21)

  Since braids can be defined by loops up to homotopy[3], a simple tug at the strands

[2]Strand indexing is relative to the position of the strands, and thus not fixed. If strand i crosses with $i+1$, strand i becomes $i+1$ and vice versa.

[3]This definition defines the braid group as $B_n = \pi_1(UConf_n(\mathbb{R}^2))$, i.e., the fundamental group of the nth unordered configuration space of $\mathbb{R}^2$. Notice that it is unordered such that the points are invariant under the symmetric group, and thus different permutations can be obtained with homotopic loops. The ordered variant results in the pure braid group P_n.

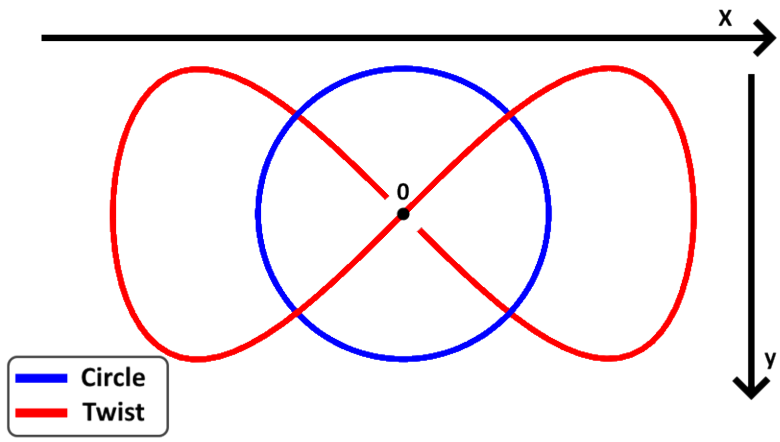


Figure 2.5: The circle piece $\mathbb{S}_1$ (blue, (2.22)) and an isotopic embedding with a twist (red, (2.23)).

ends straightens out the braid to the neutral element, the trivial braid. To this end, we have inverses for all group generators $\sigma_i^{\pm 1}$, and thus we can construct inverses for any braid $\beta \in B_n$.

□

As Figure 2.1 suggests, voice leadings may be represented by braids.

Closely related to braids, we have knots. Knots are simple closed curves in $\mathbb{R}^3$, i.e., the ends of the curve connect and there are no intersections. The three dimensions are important as we need an extra dimension to allow for over- or undercrossings without collision. If we take the two-dimensional projection (like an image) of a three-dimensional space containing knots, strands may appear to cross while they actually have a distance in between them. This is possible due to the three dimensions. However, if one has two dimensions, a crossing is an actual crossing where continuous strands collide, which is not desirable. For example, in Figure 2.5, the red curve changes in z coordinates such that around the origin the curve does not collide with itself. If however, there were no three but two dimensions, such a distance cannot be created, and a non-injective collision occurs. Hence, three dimensions are necessary for crossings.

Definition 2.37. ***Knots*** *are simple closed curves in* $\mathbb{R}^3$ *and are homeomorphic to the circle* $\mathbb{S}_1 = \{(x, y) \,|\, x^2 + y^2 = 1\}$*, as in Definition 2.19, for a knot* $K \subset \mathbb{R}^3$ *there exists an embedding* $f = \mathbb{S}_1 \to \mathbb{R}^3$ *with* $f(\mathbb{S}_1) = K$ *such that* f *is injective, continuous, and its inverse restricted to* K *is also continuous.*

Hence, collisions are not allowed since the map f would not be injective. Braids can be made into knots by connecting the end of the braid with the start of the braid as demonstrated in Figure 2.6. These are called closed braids.

The theory of knots is an interesting piece of mathematics as knots cannot be rigorously characterised or distinguished from each other. It is quite difficult to distinguish complex knots and recognise equivalent ones. An example of two equivalent knots is shown in Figure 2.8. 'Equivalent' knots are defined by ambient isotopy.

Definition 2.38. *A continuous map* $H : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ *for embeddings* f *and* g *of knots* K_1 *and* K_2 *from* $\mathbb{S}_1$ *into* $\mathbb{R}^3$ *is an* ***ambient isotopy*** *if we have*

- $H(y, 0) = Id$ *in* Y*, such that* $H(y, 0) \circ f = f$*, and*
- $H(y, 1) \circ f = g$*.*

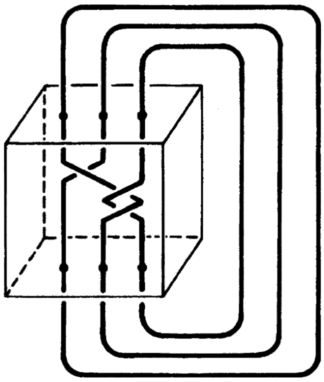


Figure 2.6: A closed braid $\sigma_1\sigma_2^{-2}$[18].

Additionally, we want that for any $t \in [0,1]$, $h_t(y)$ is again a homeomorphism. This definition is equivalent with an orientation-preserving homeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$, such that $F(K_1) = K_2$.

But it is not trivial to consider such continuous maps over $\mathbb{R}^3$. Luckily, we have the isotopy extension theorem [14, Chapter 8, Thm 1.3] that has as a consequence that a smooth ambient isotopy is equivalent with a smooth isotopy of the knot embeddings. We thus have:

Proposition 2.39. *Two knots in $\mathbb{R}^3$ are smoothly ambient isotopic if and only if the embeddings are smoothly isotopic.*

This is more useful when trying to prove equivalence of knots by their embeddings.

That is, knots K_1 and K_2 with their respective embeddings from $\mathbb{S}_1$, f and g, are isotopic if there exists a smooth isotopy $H : \mathbb{S}^1 \times [0,1] \to \mathbb{R}^3$ from the embedding of f to the embedding g, using Proposition 2.39. If these intermediate maps $h_t(\theta)$, for $\theta \in \mathbb{S}_1$, are not homeomorphisms, one could have a map where strands collide (not injective) which is physically not possible and not desirable for equivalent knots.

Example 2.40. The following map is an embedding from $[0, 2\pi]$ to the circle piece $\mathbb{S}_1$[4]:

$$\Big(\cos(\theta),\ \sin(\theta), 0\Big) \qquad \text{for } \theta \in [0, 2\pi]. \tag{2.22}$$

This is the blue circle in Figure 2.5. We can do something similar for an unknot with a twist in it, with a variable $\varepsilon > 0$ signifying the 'distance' around the twist[5]. In Figure 2.5, this is the distance around the origin for the red curve, making it such that its strand does not collide on the zero point. A value of zero for ε results in a collision, and the embedding would thus not be injective. The embedding for the red curve has the following parameterisation:

$$\Big(2\cos(\theta),\, \sin(2\theta),\, \varepsilon \sin(\theta)\Big) \qquad \text{for } \theta \in [0, 2\pi] \tag{2.23}$$

We have the following smooth isotopy H:

$$\mathbb{S}^1 \times [0,1] \to \mathbb{R}^3 : (\theta, t) \mapsto \Big((1+t)\cos(\theta),\, (1-t)\sin(\theta) + t\sin(2\theta),\, t\varepsilon \sin(\theta)\Big). \tag{2.24}$$

[4]This can also be done more trivially by the identity map $Id : \mathbb{S}_1 \to \mathbb{S}_1$. Using $[0, 2\pi]$, we assume that the image of 0 is the same as that of 2π, i.e., $f(0) = f(2\pi)$.

[5]The actual distance is $2\varepsilon > 0$.

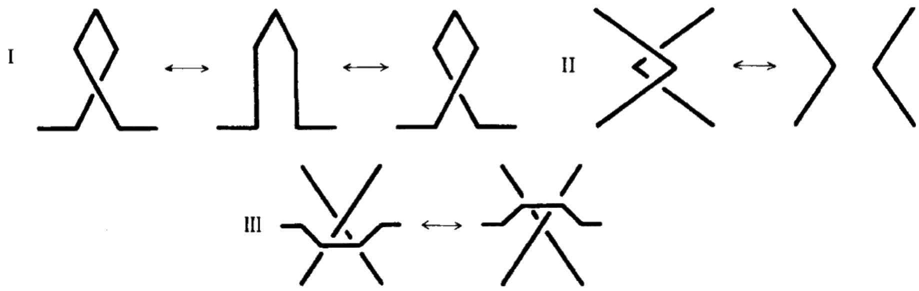


Figure 2.7: Reidemeister moves on knots. (I) displays the twist, (II) displays poke, and (III) slide.[18]

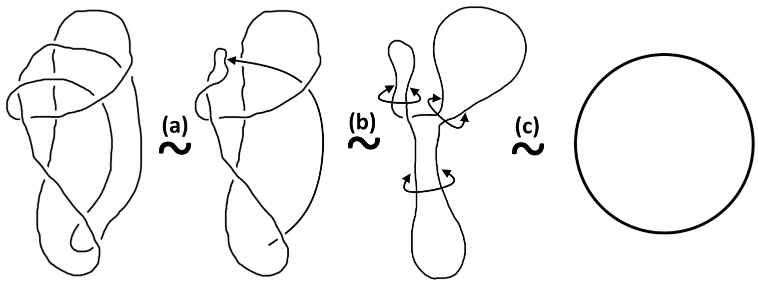


Figure 2.8: Equivalent knots. All four are the trivial 'unknot'. The equivalence is shown with Reidemeister moves. Move (a) represents three pokes. Move (b) represents three twists and move (c) deforms the knot into a circle.

We clearly have that H is a differentiable map with $H(\theta, 0)$ the embedding of the circle in (2.22) and $H(\theta, 1)$ the embedding of the twist in (2.23), and both $\mathbb{S}_1$ and $\mathbb{R}^3$ are smooth manifolds. Thus, the circle piece $\mathbb{S}_1$ is equivalent with the knot from (2.23) with a twist in it.

Visually, knots can also be made equivalent by so-called Reidemeister moves. These are three operations that can be applied to the knot such that it remains equivalent to the original knot. These operations are twist, poke, and slide, also shown respectively in Figure 2.7 and is demonstrated in Figure 2.8. A knot is thus an embedding of one circle piece $\mathbb{S}_1$, but we can also look at embeddings of multiple disjoint circle pieces. These are called links.

Definition 2.41. *A* ***link*** *is an embedding of multiple of $\mathbb{S}_1$ circle pieces into $\mathbb{R}^3$. An embedding of m disjoint circle pieces, has m components. A link basically links multiple knots together. A knot is a link with one component.*

It has been shown [18] that in general all links, and thus also all knots, can be represented as closed braids. An example of a link is the Hopf link, displayed in Figure 2.9, containing two interlinked unknots.

For braids there is an obvious surjective homomorphism M from the braiding group B_n into the permutation group of the same size S_n.

Definition 2.42. *We have a map $M : B_n \to S_n$ that maps braid generators σ_i to a permutation $(i\ i+1)$, i.e.,*

$$M : B_n \to S_n : \sigma_i \mapsto (i\ i+1), \qquad \forall 1 \leq i < n. \tag{2.25}$$

Remember from Theorem 2.12 that permutations $(i\ i+1)$ are generators of S_n, and thus we map from generators in B_n to generators in S_n. Also notice the similarities in the

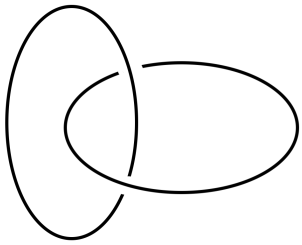


Figure 2.9: The Hopf link containing two interlinked unknots.

relations for the generators of S_n and B_n. The only difference is that in the braiding group we do not have $\sigma_i^2 = 1$.

Remark 2.43. One can show that the map M is a homomorphic map [8]. By definition, we thus also have $M(\sigma_i^{-1}) = (i\ i+1)$ for $1 \leq i < n$. By the definition of a group homomorphism, we thus have:

$$M(\beta_1\beta_2) = M(\beta_1) \circ M(\beta_2). \tag{2.26}$$

The number of components in a link depends solely on the permutation of the braid. The number of components are the number of disjunct cycles in the permutation.

A subgroup of the braid group B_n is the pure braiding group P_n.

Definition 2.44. *The* ***pure braid group*** P_n *is the group of all braids of whom the induced permutation from* M *is the identity permutation, i.e.,* $P_n = \ker(M)$.

By the homomorphism M from B_n to S_n, the pure braids are clearly a subgroup of B_n as for two pure braids γ_1 and γ_2, we have

$$M(\gamma_1 \circ \gamma_2) = M(\gamma_1) \circ M(\gamma_2) = Id \circ Id = Id. \tag{2.27}$$

This subgroup is generated by elements of the form Definition 2.42

$$A_{j,k} = \left(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_{j+1}\,\sigma_j^2\,\sigma_{j+1}\cdots\sigma_{k-1}\right), \quad \text{for } 1 \leq j < k \leq n, \tag{2.28}$$

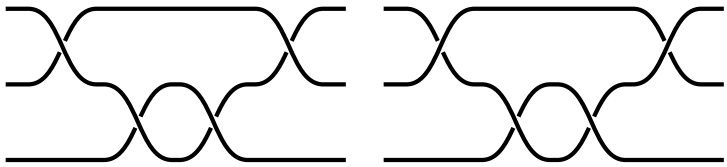
with relations:

1. $[A_{r,s}, A_{i,j}] = 1, \quad \text{for } 1 \leq r < s < i < j \leq n \text{ or } 1 \leq r < i < j < s \leq n.$
2. $A_{r,s}A_{r,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}A_{s,j}, \quad \text{for } 1 \leq r < s < j \leq n.$
3. $A_{r,s}A_{s,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{s,j}, \quad \text{for } 1 \leq r < s < j \leq n.$
4. $[A_{i,j}^{-1}A_{s,j}A_{i,j}, A_{r,i}] = 1, \quad \text{for } 1 \leq r < s < i < j \leq n,$

where $[\cdot,\cdot]$ denotes the commutator, i.e., $[g,h] = g^{-1}h^{-1}gh$.

By the first isomorphism theorem and M (Definition 2.42), we have that B_n/P_n is isomorphic with S_n [8]. This makes sense as the pure braids are in the kernel of the homomorphism M, and thus all have the identity permutation from the homomorphism M.

Example 2.45. Examples of pure braids are the following:



(2.29)

with braid words $\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1} = A_{1,3}^{-1}$ and $\sigma_2\sigma_1^{-2}\sigma_2 = A_{2,3}A_{1,3}^{-1}A_{2,3}$ respectively.

Since the actual crossing types (over and under) do not really matter for the permutation, they will also not matter for the quotient with B_n. For any pure braid, we can change any crossing, and it will still be a pure braid. Similarly, in the quotient group B_n/P_n, we will also have that crossing types do not matter.

Proposition 2.46. *In the quotient group $B_n/P_n \cong S_n$, we do not distinguish between the types of crossings, i.e., if braid words β and β' only differ in some crossing types, they will be equivalent in B_n/P_n.*

Proof. Consider a braid $\beta = \gamma\beta_1$ with γ a pure braid. Now if we take a braid β' such that β' only differs from β in the type of crossings, we know that there is a pure braid γ' and a braid β_1' such that we have $\beta' = \gamma'\beta_1'$, such that the pairs of sub-words (γ, γ') and (β_1, β_1') also differ only in crossing type. Since γ and γ', and respectively β_1 and β_1', only differ in crossing types, their permutation (from the map M as defined in Definition 2.42) is the same. Thus, we can form a pure braid $\gamma^* = (\gamma')(\beta_1')(\beta_1)^{-1}(\gamma^{-1})$ such that $\gamma^*\beta = \beta'$, making the two braids equivalent in B_n/P_n, independent on the crossing types. □

If braids are algebraically equivalent in the braiding group, they will represent the same knot, e.g. $\sigma_1\sigma_2^{-1}$, $\sigma_1\sigma_3\sigma_3^{-1}\sigma_2^{-1}$, and $\sigma_3\sigma_1\sigma_3^{-1}\sigma_2^{-1}$ in B_4. There exist also so-called Markov moves that ensure that two closed braids are isotopic if there exists a finite set of these Markov moves to transform one braid into the other[18]. The Markov moves are as follows:

1. The first move is conjugation. A braid $\beta = \beta_1\beta_2 \in B_n$ is Markov equivalent by conjugation to another braid $\beta' = \beta_2\beta_1 \in B_n$, where two parts of the braid have been switched in their order of appearance. When notating β_2 as just one generator σ_i, we can also write this equivalence as one between a braid β with $\sigma_i\beta\sigma_i^{-1}$ for $0 < i < n$ in B_n.

2. The second Markov move is stabilisation. A braid $\beta \in B_n$ is Markov equivalent to a braid $\beta\sigma_{n+1}^{\pm 1} \in B_{n+1}$. Here the last crossing does not change the knot by introducing the extra $(n+1)$th string. This move is equivalent with a twist Reidemeister move.

Stabilisation also allows us to make the following homomorphism:

$$B_n \to B_{n+1} : \beta \mapsto \beta, \tag{2.30}$$

basically embedding[6] B_n into B_{n+1} by adding another strand that has no interaction with the initial braid. This then also allows us to map any braid $\beta \in B_n$ into B_m with $n \leq m$.

We can say the following, to summarise, for knot isotopic links or knots:

Proposition 2.47. *Two links L_1, L_2 are equivalent if*

[6] Not in the sense of Definition 2.21.

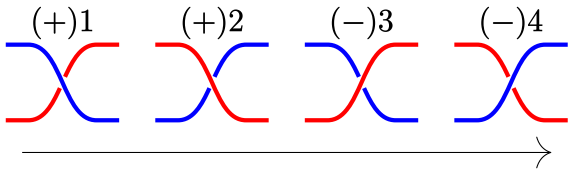


Figure 2.10: Four types of crossings that two components (red and blue) can do, following the (usual braid) orientation indicated by the arrow. Crossings of type 1 and 2 are positive crossings and 3 and 4 are negative.

- *they can be deformed into each other with finite amount of Reidemeister moves (knot isotopy),*
- *they have an algebraically equivalent braid representation (braid equivalence),*
- *their braid representations are Markov equivalent, i.e., can be deformed using a finite amount of Markov moves (knot isotopy from braidings).*

2.5.1 | Knot invariants and Seifert Surfaces

As mentioned, knots cannot be uniquely characterised. The way this is generally done is by introducing knot invariants. As the name suggests, these are elements that remain the same, invariant, for any equivalent representation of a knot. This is thus the general way to identify knots with each other. There exist many different types of invariants such as: the knot determinant, tricolorability, various polynomials such as Alexander-, Conway-, and Jones polynomials, and a general collection of invariants like the Vassiliev invariants[18].

Knots can be given an orientation to determine some properties. This entails following the curve of a knot in a certain direction. Some invariants (as the linking number in Definition 2.49) will be dependent on this orientation. We will talk about an oriented knot where needed.

For links there is an invariant called the linking number. It gives an idea how many components are intertwined with each other. the linking number can be found by looking at the type of crossings from Figure 2.10. If components are given an orientation, crossings will have orientation too, and thus a distinction can be made, as is shown in Figure 2.10. We distinguish between negative and positive crossings, or respectively over- or undercrossings.

Remark 2.48. Remark, that overcrossings have a positive exponent in the braid group, but are referred to as negative crossings. This is quite confusing. Hence, we shall refer to crossings by their exponent sign. Note that the initial assignment of positive and negative is still required for the following definition, Definition 2.49.

The definition of the linking number is then:

Definition 2.49. *The* ***linking number*** *of two oriented components K_1, K_2 is given by:*

$$l(K_1, K_2) = \frac{c_1 + c_2 - c_3 - c_4}{2}, \tag{2.31}$$

where $c_{,}$ c_2, c_3, c_4 are the number of occurrences of each type of crossing, displayed in Figure 2.10, respectively.

Here, we consider either K_1 blue and K_2 red, or vice versa, a distinction needs to be made. The crossings of type 1 and 2 are also referred to as the positive crossings, and 3 and 4 as negative crossings [20]. Notice that an undercrossing (bottom under top strand) is thus positive, and an overcrossing (bottom over top strand) is negative. The distinction can be made by using the right-hand rule:

Strands follow the direction of your thumb, and your other fingertips. Facing your palm, the thumb strand is on top, and facing the back of your hand the other fingers are on top. Now it is possible to mimic types 1 and 2 with your right hand, and three and four with your left hand.

The linking number is always a whole number because $c_1 + c_3 = c_2 + c_4$, and this is because two simple closed curves will always cross each other an even number of times.

For braids with fixed strands n, the exponent sum can be proven to be invariant [8]. This is the sum of all exponents of the generators in a braid $\beta \in B_n$.

Definition 2.50. *The **exponent sum** of a braid $\beta = \sigma_{i_1}^{e_1} \dots \sigma_{i_k}^{e_k} \in B_n$, with $i_j \leq n-1$ for all $1 \leq j \leq k \in \mathbb{N}$, is the map*

$$\mathrm{Exp} : B_n \to \mathbb{Z} : \beta \mapsto \sum_{j=1}^{k} e_j. \tag{2.32}$$

Notice that the map Exp is a group homomorphism, and because of this, it is a braid invariant in B_n for braids with a fixed n strands. For a proof of this statement, one may refer to [8, p43].

Another interesting invariant is the knot genus. The genus is a property of a connected, orientable surfaces that signifies the number of 'holes', or handles. The genus g can also be found using the Euler characteristic χ for a closed surface:

$$\chi = 2 - 2g. \tag{2.33}$$

Generally, the Euler characteristic of a polyhedron is given by

$$\chi = V - E + F, \tag{2.34}$$

with V the number of vertices, E the number of edges, and F the number of faces.

We can make the connection between knots and surfaces by defining Seifert surfaces:

Definition 2.51. *For a given link L, a **Seifert surface** is a compact, connected orientable surface with the link L as its boundary.*

We can give an example of a Seifert surface for a trefoil knot. The trefoil knot is the smallest nontrivial knot. It is shown in Figure 2.11. A Seifert surface for the trefoil knot is shown in Figure 2.12. Seifert surfaces are by no means unique. There exists an algorithm, Seifert's algorithm, which is deterministic and finds a Seifert surface for any given link. Seifert's algorithm can be extended to braids and is provided in Algorithm 1.

An example of the Seifert's algorithm is shown in Figure 2.13 for braids σ_1^3 and $\sigma_1^3\sigma_1^{-1}\sigma_1$.

We can now consider the genus of a Seifert surface. As in article [23], the Euler characteristic of a Seifert surface s is given by

$$\chi_s = 2 - 2g_s - m, \tag{2.35}$$

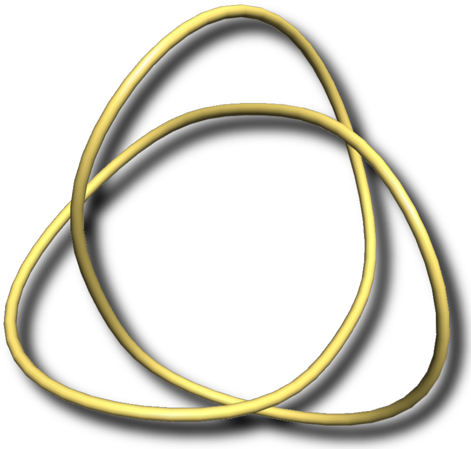


Figure 2.11: The trefoil knot. The smallest nontrivial knot. It has braiding representation σ_1^3

Algorithm 1 Seifert's Algorithm

Input: braid $\beta \in B_n$
Output: Valid Seifert surface for β
1: For every strand in the braid $\beta \in B_n$, associate a disk to it. These disks are called Seifert circles.
2: A crossing σ_i between strands at positions i and $i+1$ becomes a twisted strip connecting the ith and $(i+1)$th disks. The direction of the twist depends on the type of crossing.

with g_s its genus and m the number of knot components. For a Seifert surface coming from Algorithm 1 with c Seifert circles and d twists, we have

$$g_s = \frac{2 + d - c - m}{2}. \tag{2.36}$$

The genus of a Seifert surface can also be found by considering a closed surface. Article [23] shows a way to create closed Seifert surfaces such that a knot on the surfaces divides the surface into two equal sized parts, with Euler characteristic $\chi_c = 2\chi_s$. The genus of the Seifert surface g_s is then related to the genus of the closed surface g_c by using (2.33) and (2.35). This results in the following:

$$g_c = 2g_s + m - 1, \tag{2.37}$$

with m the number of components in a knot. Because the Seifert surfaces are not unique, it can create some redundant complexities. As an example, the Seifert surface of a trefoil knot can get higher genus than necessary. The genus invariant of a trefoil knot is one. An example is shown in Figure 2.14, where two closed surfaces have genus 2 and 4 respectively resulting in Seifert surfaces genuses of 1 and 2 respectively, for the trefoil knot. Thus, the genus invariant for a knot is then defined as the minimal genus of all Seifert surfaces of a knot. The unknot clearly has genus invariant equal to zero since a disk fits the description of a Seifert surface, and a disk can be constructed with 1 vertex (point on circle), 1 edge (the circle), and 1 face (the disk). By formula (2.35) the genus is zero, since the unknot has 1 component and Euler characteristic 1. This value is clearly minimal for the unknot.

With this preliminary knowledge, we can now start looking to some ways to use braids as representations for voice leadings.

Figure 2.12: Two Seifert surfaces for the trefoil knot with genus 1 and 2 respectively, generated with SeifertView[23] with braid representations of σ_1^3 (AAA) and $\sigma_1^3\sigma_1^{-1}\sigma_1$ ($AAAaA$). These surfaces are the smoothed version of those in Figure 2.13.

Figure 2.13: Two Seifert surfaces of the trefoil knot as output of Seifert's algorithm, containing disk and connecting twisted strips. The surfaces are generated with SeifertView[23] with braid representations of σ_1^3 (AAA) and $\sigma_1^3\sigma_1^{-1}\sigma_1$ ($AAAaA$). These are the unsmoothed versions of surfaces in Figure 2.12.

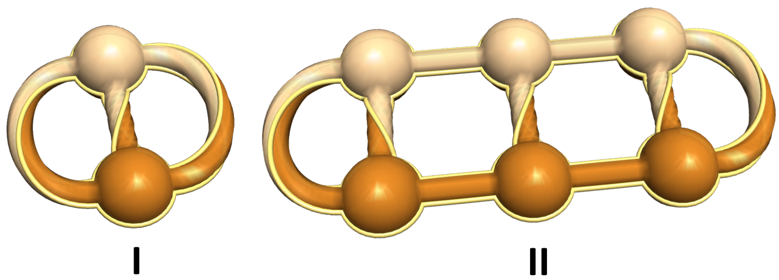


Figure 2.14: Two closed surfaces containing the trefoil knot, generated with SeifertView[23]. The first (I) surfaces clearly has genus 2 and the second (II) has genus 4

3 | Naive Braiding Representations

In this chapter, we look at some naive braiding representations as suggested by literature. We discuss a way to represent voice leadings, with voice crossings as braid crossings, and another way to represent general chord progressions. The first representation is more naive since the thesis [6, Part 4] suggests using partial singular braids, and we implement it in the regular braid group.

We will discuss some shortcomings and inconsistencies like dependencies on ordering voice leadings or having an order on chord changes. With the latter being superficial since in chord progressions, we assume all changes to be simultaneous, and thus choosing an order on these changes is unwanted.

Let us start by considering a representation for voice leadings where we will find a lack of distinction between types of crossings.

3.1 | Voice Leadings as Braids

The thesis [6] suggests a representation for voice leadings with voice crossings using partial singular braids. Before we talk about those in Chapter 6, let us try to consider this representation as regular braids first. We can construct a braid as follows, as suggested by the literature.

Let us consider a voice leading with n voices. We can create a braid in B_n by applying the algorithm given in Algorithm 2. Note that by consistently updating π in the algorithm, a voice crossing will always occur between neighbouring strands as strands that do not neighbour cannot cross without crossing others first.

Algorithm 2 Braid Voice Leadings

Input: Voice leading with n voices
Output: A braid $\beta \in B_n$

1: A voice initialises a strand i, such that neighbouring strands start on neighbouring notes of this voice.
2: We keep track of the permutation of the voices $\pi \in S_n$, such that we can still identify voices in the braid. The position in the braid of voice i is then given by $\pi(i)$. This is essentially the permutation coming from M in Definition 2.42. Initially π is the identity element.
3: The resulting braid $\beta \in B_n$ is initialised with $\beta = 1$. We will leave open the type of crossing (under or over) for now and denote a crossing as $\sigma_i^{\pm 1}$ in the braiding group.
4: When a voice crossing occurs between voices i and j, recall Definition 2.3, change β to $\beta * \sigma^{\pm 1}_{\min(\pi(i),\pi(j))}$. We update the permutation using M.

Let us have a close look at this suggested representation from [6].

This representation requires a consistent way of defining under and over crossings such that the distinction is meaningful. For now, to consider some properties, we will just apply a positive crossing (see Remark 2.48) for all voice crossings.

The voice leading example from Chapter 2 in Figure 2.1 gives then braid word σ_2,

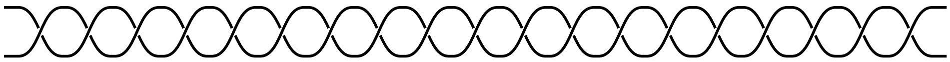
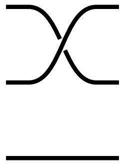


Figure 3.1: Bach piece from Appendix A made into a braid with uniform crossing type (positive).

which is the following braid:



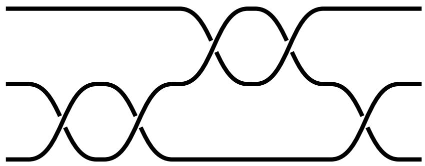
(3.1)

Note that in the voice leading in Figure 2.1, voice 2 and voice 3 actually collide but do not cross. Since, we cannot discuss such singularities (yet, see Section 6.2) for regular braids in B_n, we shall consider this as regular non-crossing, non-colliding strands. Hence, the braid word is just σ_2.

It is clear that if a voice leading does not contain any crossings, it all is reduced to the trivial braid and thus the unknot.

Example 3.1. A more real-life example is given by the snippet of Bach's concerto for two violins in Appendix A. The voices sometimes collide into one note, i.e. voices hit the same note at the same time. This is again essentially a singular point in a braid. This makes it more difficult to consider normal knot invariants to analyse this piece[7]. We again choose to ignore such points where voices bounce of each other, and assume a crossing where voices eventually cross after a collision. However, we will use the partial singular braid monoid that does distinguish between these crossings in Chapter 6. With this idea, the representing braid of this music piece has 19 meaningful strand crossings, which is of course the same number as the number of voice crossings. The braid looks like the one displayed in Figure 3.1.

Example 3.2. Consider a snippet of *Mon Chier Amy*, containing three voices, provided in Appendix B. This will give the braid $\sigma_1^2\sigma_2^2\sigma_1$ using only positive crossings. It is the following the braid:



(3.2)

Closing this braid results in a trefoil knot, linking with a second component just once. It is displayed in Figure 3.2.

It is clear that the braid provided in Figure 3.1 from Example 3.1 is solely dependent on the amount of voice crossings that occur. Especially, since we assumed all the crossing types to be uniform. Hence, when looking into knot invariants like the genus we get that it is directly dependent on the number of crossings. This is clear from the propositions in Appendix C. These results are not a surprise though, as we can prove the more general[8] idea:

Proposition 3.3. *The braid group of two strands B_2 is isomorphic (recall Definition 2.19) with $\mathbb{Z}$.*

[7] There exists extensions of knot invariants to partial singular braids, such as the Burau representations [11, 9, 11] and the already mentioned Vassiliev invariants for singular braids.

[8] General in the sense that we allow multiple types of crossings.

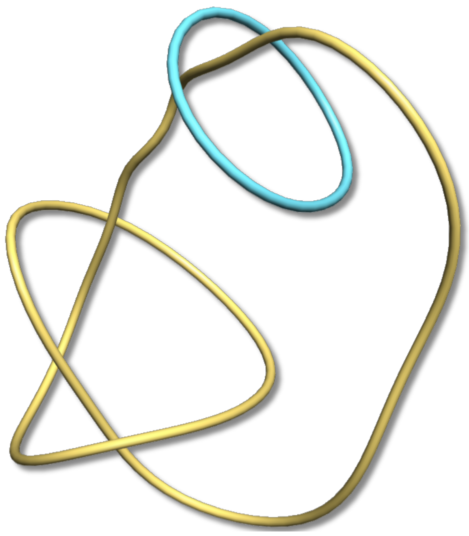


Figure 3.2: The knot resulting from closing the braid in Example 3.2. It is a trefoil knot linked with an unknot.

Proof. Recall from Definition 2.50 that we have a surjective group homomorphism, the map Exp, from B_n to $\mathbb{Z}$. What is left to show is that it also is injective for B_2, i.e., $n = 2$.

Note that σ_1 is the only generator in B_2. If we have two different braids β_1 and β_2 in B_2 such that $\text{Exp}(\beta_1) = \text{Exp}(\beta_2)$, we have the existence of some extra negative and positive generators such that the sum still matches, i.e., there exists a $k \in \mathbb{N}$ such that β_2 contains k extra σ_1^{-1} generators and k extra σ_1 generator. But since this is the only generator these cancel each other in the braid group, concluding $\beta_1 = \beta_2$ in B_2. Thus, the map $\text{Exp} : B_2 \to \mathbb{Z}$ is a bijective homomorphism, i.e., an isomorphism. □

The map from $\mathbb{Z}$ to B_2 is the following one:

$$\mathbb{Z} \to B_2 \ : \ z \mapsto \sigma_1^z. \tag{3.3}$$

Thus it makes sense that the number of crossings immediately influences the braid's properties such as the genus of Seifert surfaces in B_2.

With Example 3.2, we clearly have some more complexity since we now have 3 strands crossing each other. This is actually a solid presentation. However, to fully exploit the braiding group, we need to distinguish between crossings. If we do not distinguish between crossings, we will have no need for braids, as we will see in Corollary 4.3.

Since a musical piece with voice crossings does not distinguish between different types of crossings, a braid defined with monotone crossings might not introduce any meaningful complexity to characterise the musical piece. We will continue this idea in Chapter 4.

Another issue with this approach is that, if the voices are not already defined, there will always be a permutation in notes for the voices such that the voices do not cross. For example, in Figure 2.1, one could have voice 3 go to G_4 and voice 2 to E_4 to avoid crossings. For chord progressions, this is not very useful. The braids are defined based on the voice crossings, which in turn depend on the permutation of the chords, and this can be trivial.

For chord progressions, the problem of trivial permutations is an issue since chord progressions do not define distinct voices because all changes happen simultaneously. Thus, this representation is definitely not fit for chord progressions.

Also note that with more strands in a braid, it will be harder to find music pieces with n voices that cross, since a meaningful braid in B_n requires that all the n strands cross with each other. Thus, one needs n voices that all cross, and thus all the voices are in the same frequency domain, e.g., low frequencies, middle, or high frequencies.

3.2 | Chord Progressions as Braids

In article [5], another way is introduced to model chord progressions with braids. We look into the characterisation here and discuss it.

This time, the braid group is always B_{12} for 12 different pitch classes. We denote the space of pitch classes as follows:

Definition 3.4. *The* ***space of pitch classes*** $\mathbb{T}$ *is defined as the quotient of the octave equivalence over all pitches in* $\mathbb{R}$*, i.e.*

$$\mathbb{T} = \mathbb{R}/12\mathbb{Z} = \overline{[0, 12)}. \tag{3.4}$$

This is a circle.

Obviously, not all (continuous) pitches can be represented by a finite amount of strands, and thus, the braid representation will only consider all 12 notes from (2.1). To consider chords consisting of n notes, we can consider the n-fold Cartesian product $\mathbb{T}^n$. However, this structure has an embedded ordering, e.g. (G, B) is not the same element as (B, G) in $\mathbb{T}^2$. For chords in general, we want to consider them as similar elements. That is why we also take the permutation equivalence out of the equation with the following chord space:

Definition 3.5. *The* ***chord space*** $\mathbb{A}_n$ *contains unordered* n*-tuples of pitch classes* ($\in \mathbb{T}$)*, i.e.*

$$\mathbb{A}_n = \mathbb{T}^n/S_n, \tag{3.5}$$

with S_n *the symmetric group, as in Definition 2.10.*

To give an example, the $\mathbb{T}^2$ space is a donut and $\mathbb{A}_2$ is the Möbius band.

Remark 3.6. We will consider non-singular elements of the higher-dimensional chord space, i.e., $\mathbb{A}_n$ with $n \geq 2$. Non-singular chords are those that do not repeat notes, i.e., they have n distinct pitches. Otherwise, a chord in $\mathbb{A}_n$ with less than n distinct pitches is more like a chord in the smaller space $\mathbb{A}_m$, with $m \leq n$ and m distinct pitches.

The construction suggested in dissertation [5] also uses voice leadings, but uses 12 strands to model the braid and voice crossings are specifically avoided. For two chords in $\mathbb{A}_n$ with pitches $(p_1, \ldots, p_n)$ and $(q_1 \ldots, q_n)$ of equal size, the braid is constructed using Algorithm 3.

Example 3.7. Let us consider the chord progression from C^{maj7} to E^7 to A^7 to Dm^7, the intro to jazz tune *All of Me*. The chords have the following notes:

$$\begin{aligned} C^{maj7} &= (C, E, G, B), & E^7 &= (E, G^\sharp, B, D), & A^7 &= (A, C^\sharp, E, G), & Dm^7 &= (D, F, A, C), \\ &= (0, 4, 7, 11), & &= (4, 8, 11, 2), & &= (9, 1, 4, 7), & &= (2, 5, 9, 0). \end{aligned} \tag{3.7}$$

Algorithm 3 Braid Chord Progressions

Input: A chord progression

Output: A braid $\beta \in B_{12}$

1: Voice crossings are avoided by imposing the condition:

$$p_{v_i} > p_{v_j} \implies q_{v_i} \geq q_{v_j}, \tag{3.6}$$

with v_i and v_j denoting two distinct voices. Voice leadings are easily constructed by ordering the chords based on their pitch class and making voice leadings based on that order. For example, the chords C^{maj7} and G^7 can be represented by $(0, 4, 7, 11)$ and $(7, 11, 2, 5)$ respectively. Thus, the resulting voice leadings are $0 \mapsto 2$, $4 \mapsto 5$, $7 \mapsto 7$, $11 \mapsto 11$.

2: In all voice leadings, if we have a voice going $k \mapsto l$ with $k < l$ in $\mathbb{T}$ (recall Definition 3.4), we will represent this by a braid $\sigma_{k+1}\sigma_{k+2}, \ldots, \sigma_l$. If the voice leading $k \mapsto l$ has $k > l$ in $\mathbb{T}$, we apply braid $\sigma_k\sigma_{k-1}, \ldots, \sigma_{l+1}$. If a voice stays constant, we apply no braid, or equivalently the neutral element 1. For example, $0 \mapsto 2$ becomes $\sigma_1\sigma_2$, and $2 \mapsto 0$ becomes $\sigma_2\sigma_1$.

Here, we used C as zero, but this is an arbitrary choice. By Algorithm 3, the chosen voice leading is the following:

$$\begin{aligned} 0 \mapsto 2 \mapsto 1 \mapsto 0, \\ 4 \mapsto 4 \mapsto 4 \mapsto 2, \\ 7 \mapsto 8 \mapsto 7 \mapsto 5, \\ 11 \mapsto 11 \mapsto 9 \mapsto 9, \end{aligned} \tag{3.8}$$

where we use the notation from Definition 2.1. These voice leadings are displayed in Figure 3.3.

We can thus form the following sub-words per chord change:

- C^{maj7} to E^7: $\sigma_1\sigma_2\sigma_8$,
- E^7 to A^7: $\sigma_2\sigma_8\sigma_{11}\sigma_{10}$,
- A^7 to Dm^7: $\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6$.

Thus, the total the braid is $\sigma_1\sigma_2\sigma_8\sigma_2\sigma_8\sigma_{11}\sigma_{10}\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6$, displayed Figure 3.4.

Again, we have a closer look at this representation. There are a couple of issues with this setup. Note that it is not invariant under modulation, i.e. if the chord progression is transposed to another key, the braid changes. Equivalently, if we choose to start counting in $\mathbb{T}$ on another note than C, e.g. G becomes 0, the braid changes. To see this, take the map $0 \mapsto 11$, from C to B. Since we have taken octaves out of the question this is the same map as $0 \mapsto -1$, and thus if we transpose this up by $k < 12$ steps we simply get a jump of 1, instead of a jump from 0 to 11. Hence, their respective braids are $\sigma_1 \ldots \sigma_{11}$ and another braid which will simply be one generator σ_{k+1} for the transposition of k steps up.

This problem also occurs with the condition (3.6) in Algorithm 3, which is quite ill-defined considering pitches in $\mathbb{T}$ (recall Definition 3.4). Take for example the following chord progression $(0, 6) \rightarrow (5, 11)$[9]. To avoid collisions, one takes maps $0 \mapsto 5$ and $6 \mapsto 11$,

[9] C augmented power chord to F diminished power chord.

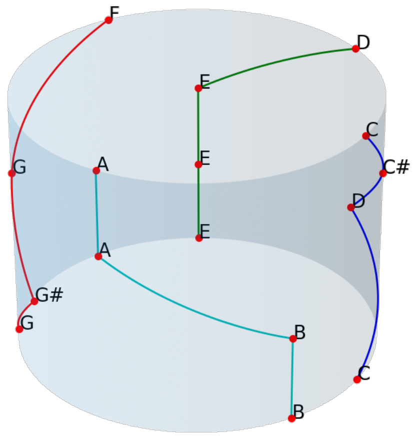


Figure 3.3: The voice leading from Example 3.7 displayed in fashion with [6], i.e., represented on a cylinder that represents the pitch space $\mathbb{T}$ over time.

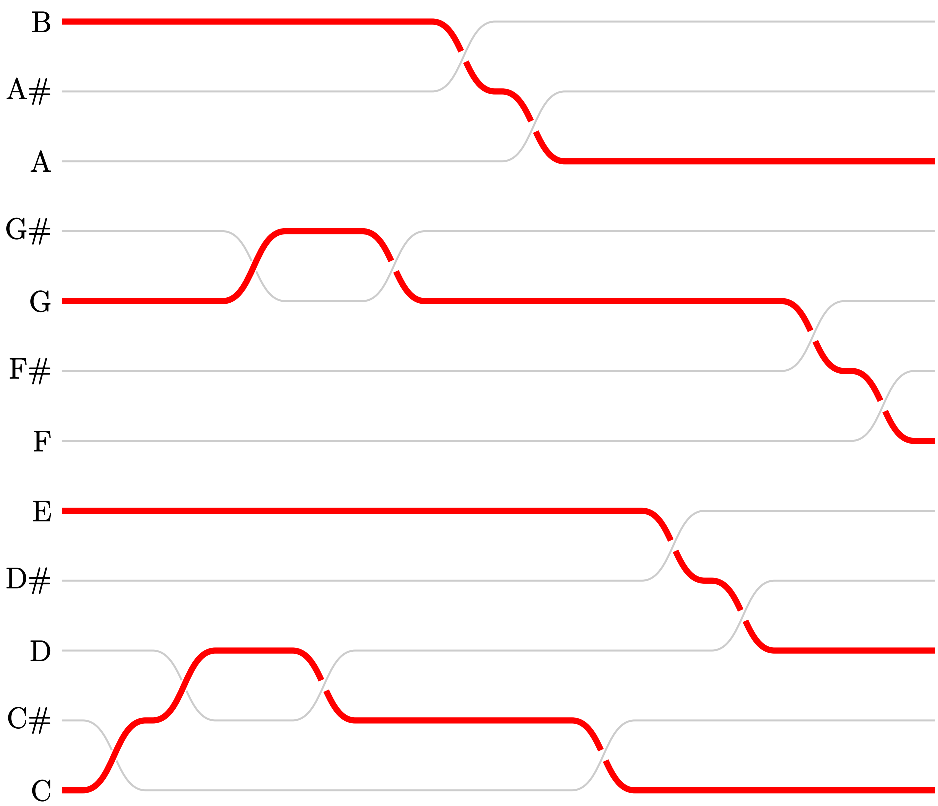


Figure 3.4: Chord progression from *All of Me* as described in Example 3.7.

resulting in a very inflated braid $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}$. But taking maps $0 \mapsto 11$ and $5 \mapsto 6$ is a much more natural way to define the voices. Here, the map $0 \mapsto 11$ will again inflate the braid as previously discussed.

However, we will come to find out later in Chapter 4, that a base always has to be chosen, and thus not being invariant under modulation is not the worst.

We can finetune the braid to account for the octave equivalence of pitch classes in the torus $\mathbb{T}$. As is shown in the papers [16, 12, Thm1, Prop2.1 resp.], we can define a natural metric on the quotient space. The natural metric on $\mathbb{T}^n$ is

$$d_{\mathbb{T}^n}(x, y) = \inf_{k \in \mathbb{Z}^n} \|x - (12k + y)\|, \tag{3.9}$$

where $\|\cdot\|$ is a norm defined on $\mathbb{R}^n$. Similarly, we can define a metric on the chord space $\mathbb{A}$:

$$d_{\mathbb{A}_n}(x, y) = \inf_{\pi \in S_n} d_{\mathbb{T}^n}(x, \pi(y)). \tag{3.10}$$

When we equip $\mathbb{R}^n$ with the L_1 norm, we get:

$$d_{\mathbb{A}_n}(x, y) = \inf_{\pi \in S_n} \inf_{k \in \mathbb{Z}^n} \|x - (12k + y)\|_1. \tag{3.11}$$

This makes for a natural way to define the voice leadings, i.e. with the permutation $\mathrm{argmin}_{\pi \in S_n} d_{\mathbb{T}}(x, y)$. The concept of the 'distribution constraint'[13] is baked into this definition, meaning that the resulting permutation will always give the most efficient, non-crossing permutation, independent of the chosen norm on $\mathbb{R}^n$. We can consider such a voice leading to be 'minimal'. All p-norms will give the same permutation[13]. This will sort out the issue we pointed out with the example $(0, 6) \to (5, 11)$.

Another issue with the method in Algorithm 3, is that the order of voice changes should not matter. As mentioned previously, when considering chord progressions, all changes occur simultaneously and thus no order is required.

Each voice leading generates its own sub-word, and since the order of appearance does not matter musically, we want this as well for the braids. Optimally, the strands representing an individual voice do not interact (cross) and we can generally switch these components around by the second braid relation in (2.19). Explicitly, for a braid $\beta = \beta_1\beta_2$ we have:

$$\forall \sigma_i \in \beta_1{}^{10}, \forall \sigma_j \in \beta_2 : \ |i - j| \geq 2 \Rightarrow \beta = \beta_2\beta_1. \tag{3.12}$$

However, we can find examples where this is not the case. Namely, we can find examples of chord progressions with minimal voice leadings such that the order of the sub-words does matter. If we take for example a progression going from $(0, 1, 2)$ to $(2, 3, 4)$, with minimal, non-crossing voice leading $0 \mapsto 2$, $1 \mapsto 3$, and $2 \mapsto 4$, we have the following braid words representing each voice leading.

- $0 \mapsto 2$ becomes $\sigma_1\sigma_2 =: \beta_1$,
- $1 \mapsto 3$ becomes $\sigma_2\sigma_3 =: \beta_2$,
- $2 \mapsto 4$ becomes $\sigma_3\sigma_4 =: \beta_3$.

Since we have modulated out permutations, the order should not matter. Also, the order of these voice leadings in the music is arbitrary since all changes occur simultaneously.

[10] For ease of notation, let us consider a generator an element if it occurs in the respective braid word.

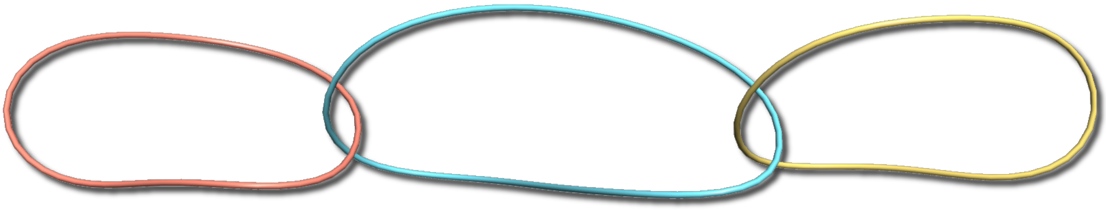


Figure 3.5: A chain linking 3 components from the braid word $\sigma_1\sigma_2^2\sigma_3^2\sigma_4$. It is isotopic with the braid $\sigma_1^2\sigma_2^2$.

By the Markov equivalence moves, we know that a braid of the form $\beta_1\beta_2\beta_3$ is equivalent with $\beta_3\beta_1\beta_2$, and $\beta_2\beta_3\beta_1$. This equivalence does not include equivalence with $\beta_3\beta_2\beta_1$ and its Markov equivalents $\beta_1\beta_3\beta_2$ and $\beta_2\beta_1\beta_3$. Thus, these are not per se equivalent. For example, the order $\sigma_1\sigma_2^2\sigma_3^2\sigma_4$ gives 2 unknots that are linked together by a third component (shown in Figure 3.5). The braid $\sigma_1\sigma_2^2\sigma_3^2\sigma_4$ is knot-isotopic with $\sigma_1^2\sigma_2^2$. The other order, $\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_4$, gives the trefoil knot. We can show the latter Markov equivalence as follows:

1. $\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_4$ is Markov equivalent (by stabilisation) to $\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$,
2. $\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$ is Markov equivalent to $\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2$ by Markov conjugation,
3. $\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2$ is algebraically equivalent to $\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2$.
4. Again using Markov stabilisation (and conjugation) we can remove the sole σ_3 generator. We get $\sigma_2\sigma_1\sigma_2\sigma_2$.
5. This is algebraically equivalent to $\sigma_1\sigma_2\sigma_1\sigma_2$.
6. By conjugation we get $\sigma_2\sigma_1\sigma_2\sigma_1$, which in turn becomes $\sigma_1\sigma_2\sigma_1\sigma_1$.
7. Now again, we can remove the sole σ_2 operator to conclude σ_1^3 which is the minimal braid representing the trefoil knot (shown in Figure 2.11).

Hence, this arbitrary choice in order of the sub-words makes a significant difference, and thus any complexity introduced using this way is very artificial, in the sense that it has no added natural meaning.

In the next chapter we will try to show that a braid representation is therefore not interesting to assume since any complexity is artificially arbitrary and any other information retained does not come from braidings.

4 | General Unordered Representation

Now that we have identified some key issues with the naive representation of the voice leadings and chord progressions with braids, we can start looking into why these occur. We will look into what kind of structures these musical elements are and how they embed into the braiding group. Considering chords, or equivalently a set of notes for voice leadings, X and Y, we will find many isomorphisms between the group of permutations on either X or Y and the set of all possible voice leadings between X and Y. Note that a group like $S(Y)$ also induces many non-canonical isomorphisms with S_n, where elements are ordered. In the representation in Section 3.2, we considered chords to be unordered, and thus there are numerous ways to reintroduce an order upon the chords, because all the orders are equivalent. Let us start by orienting ourselves within chords $\mathbb{A}_n$ and consider permutations.

Considering the natural metric on the chord space $\mathbb{A}_n$, mentioned in (3.10), we note that the induced permutation coming from this natural metric is not meaningful to consider in the space $\mathbb{A}_n$ where there is no ordering of notes.

Given 2 chords X and Y in $\mathbb{A}_n$, consider two ordered class representatives $\bar{x}_1 \in X$ and $\bar{y}_1 \in Y$, i.e., $\bar{x}_1$ and $\bar{y}_1$ have orders. Now, the metric induces a permutation on $\bar{y}_1$, dependent on $\bar{x}_1$, i.e.,

$$\pi_1^* = \text{argmin}_{\pi \in S_n} d_{\mathbb{T}^n}(\bar{x}_1, \pi(\bar{y}_1)). \tag{4.1}$$

The permutation π_1^* is here completely dependent on both $\bar{x}_1$ and $\bar{y}_1$. We can take other representatives $\bar{x}_2 \in X$ and $\bar{y}_2 \in Y$. Then, there exists permutations $\pi_x, \pi_y \in S_n$ with

$$\bar{x}_1 = \pi_x(\bar{x}_2) \qquad \bar{y}_1 = \pi_y(\bar{y}_2). \tag{4.2}$$

Obviously, $\bar{x}_2$ and $\bar{y}_2$ also induce a permutation in the same way, notate π_2^*. We then have the following

$$\pi_2^* = \pi_x \circ \pi_1^* \circ \pi_y^{-1}. \tag{4.3}$$

Basically, we are removing the order equivalence of $\mathbb{A}_n$ in (4.1) and noting that order matters. Hence, to consider permutations in this context, we require a base representative. It makes more sense to consider permutations in the equivalent group $S(Y)$ (recall Definition 2.10), which has multiple isomorphism to S_n dependent on the chosen representatives $\bar{y}$. The group $S(Y)$ are bijections from Y to itself, but we require maps from X to Y. Hence, we consider the following definition.

Definition 4.1. *For non-singular chords (recall Remark 3.6) X, $Y \in \mathbb{A}_n$, the* ***set of chord progressions*** *is the set of all bijections with domain X and codomain Y:*

$$\mathcal{P}(X,Y) = \left\{p : X \to Y \mid p \textit{ is a bijection}\right\}, \tag{4.4}$$

in accordance with Definition 2.1.

The set $\mathcal{P}(X,Y)$ is clearly related to the symmetric group $S(Y)$, since the set $\mathcal{P}(X,Y)$ basically contains all permutations of X to Y. We shall prove this more explicitly. We can take an arbitrary base progression $p_0 : X \to Y$ in $\mathcal{P}(X,Y)$, for example from class representations $\bar{x}_0 \in X$, $\bar{y}_0 \in Y$ such that $\bar{x}_0$ and $\bar{y}_0$ are ordered, i.e., they are elements of $\mathbb{T}^n$ that represent their equivalent class in $\mathbb{A}_n$. Such elements can induce a chord progression, but we will talk about this in Section 5.3. Now we shall prove that all maps in $\mathcal{P}(X,Y)$ can be written as $\pi \circ p_0$ for $\pi \in S(Y)$ for such an arbitrary base progression p_0.

Theorem 4.2. *Given a base chord progression $p_0 \in \mathcal{P}(X,Y)$, with X, $Y \in \mathbb{A}_n$ non-singular chords, all other chord progressions $p \in \mathcal{P}(X,Y)$ can be formed using permutations $\pi \in S(Y)$, i.e.,*

$$\forall p \in \mathcal{P}(X,Y),\ \exists!\ \pi \in S(Y):\ p = \pi \circ p_0 \tag{4.5}$$

A similar result can be shown for $\pi \in S(X)$ and the composition $p_0 \circ \pi$.

Proof. We shall prove this for $S(Y)$. The result for $S(X)$ is analogous. Note that since all permutations are by definition bijective, and p_0 as well by definition, we have that the composition $\pi \circ p_0$ will also be bijective. Also note that the domain and codomain of the composition match those of $\mathcal{P}(X,Y)$. Thus, we have shown one inclusion, that of the set of compositions with $S(Y)$ in $\mathcal{P}(X,Y)$.

For the other inclusion note that p_0 can trivially be formed by composing with the identity permutation. For any other given $p \in \mathcal{P}(X,Y)$, we shall prove that the map from $p_0(X)$ to $p(X)$ is bijection with domain and codomain $Y \to Y$ respectively, such that it is in $S(Y)$.

Both images $p_0(X)$ and $p(X)$ are Y because the mappings are bijective. We make such a map $m : Y \to Y$ as follows:

For every element $x \in X$, map $p_0(x)$ to $p(x)$, i.e., $m(p_0(x)) = p(x)$.

Notice that the notation is well-defined because of the bijection of p_0 and p. By uniqueness of p_0 and p, m is also unique. Now clearly, $p = m \circ p_0$, and with both p_0 and p being a bijection, so is m. Thus, we have $m \in S(Y)$. □

This way the group $S(Y)$ acts on the right on $\mathcal{P}(X,Y)$, by Definition 2.15. Similarly, $S(X)$ acts on the left on $\mathcal{P}(X,Y)$. Note that there is only one orbit on X from any base element like $p_0 \in \mathcal{P}(X,Y)$, by Definition 2.17, and the action is thus transitive. Also note that the concatenation of any non-trivial element π in $S(Y)$ (respectively $S(X)$) with p, resulting in $\pi \circ p$, is different from p, and thus the action is also a free action. By Definition 2.18, we have that $\mathcal{P}(X,Y)$ is a left $S(Y)$-torsor and a right $S(X)$-torsor. This relation is independent of any initial progression p_0, but each p_0 will give a different bijection $S(Y) \to \mathcal{P}(X,Y)$. The result from Theorem 4.2 allows us to create non-canonical isomorphisms to $S(Y)$ (respectively $S(X)$). Consider the following map m:

$$m : \mathcal{P}(X,Y) \times \mathcal{P}(X,Y) \to S(Y) : (p_0, p) \mapsto \pi, \qquad \text{such that } p = \pi \circ p_0. \tag{4.6}$$

The map m is well-defined as the direct result of Theorem 4.2. With this map, we can create more isomorphisms as follows:

$$m_{p_0} : \mathcal{P}(X,Y) \to S(Y) : p \mapsto \pi, \qquad \text{such that } p = \pi \circ p_0. \tag{4.7}$$

We can consider a group operation on $\mathcal{P}(X,Y)$, dependent on p_0, from the map in (4.6):

$$*_{p_0} : \mathcal{P}(X,Y) \times \mathcal{P}(X,Y) \to \mathcal{P}(X,Y) : (p_1, p_2) \mapsto \pi_1 \circ \pi_2 \circ p_0, \tag{4.8}$$

with that $p_1 = \pi_1 \circ p_0$ and $p_2 = \pi_2 \circ p_0$. This thus allows us to form multiple group isomorphisms between $(\mathcal{P}(X,Y), *_{p_0})$ and $(S(Y), \circ)$ for all $p_0 \in \mathcal{P}(X,Y)$.

With this we can also consider an isomorphism with S_n by choosing a base progression $p_0 \in \mathcal{P}(X,Y)$, and an ordering the elements of X (or Y), e.g., making p_0 the identity permutation with a certain ordering. Or one can also order both X and Y, as will be done in Chapter 5, to get the actual permutations from S_n. But again these chosen orderings are merely class representative, and by no means unique. This introduces again many different isomorphisms, more than the ones we had from $\mathcal{P}(X,Y)$ to $S(X)$ or $S(Y)$.

To get an isomorphism to S_n, as mentioned, one first need to introduce an ordering to Y (respectively X), i.e., a bijection map $b : Y \to \{1, \ldots, n\}$, such that the following map:

$$s : S(Y) \to S_n : \pi \mapsto b \circ \pi \circ b^{-1}, \tag{4.9}$$

is a group isomorphism. we can conclude that there are multiple different non-canonical isomorphism between S_n and $\mathcal{P}(X, Y)$.

4.1 | Braid Representation

Now that we have some idea of what the set $\mathcal{P}(X, Y)$ looks like, we can try to embed it into B_n, as that is the main idea of this thesis. Since we have a map M (Definition 2.42) which makes it that any braid induces a permutation, we have that S_n is strictly smaller in structure than B_n. As a consequence, we will find many possibilities for lifting elements of $\mathcal{P}(X, Y)$ to B_n.

Note that in both representations from Chapter 3, we have that our presentation is solely defined on either over or under crossings, i.e. only one type of crossing. We could then easily define the same construction with using only the other type of crossing, e.g., $\sigma_1\sigma_2$ becomes $\sigma_1^{-1}\sigma_2^{-1}$. This results in an equivalence relation $\sigma_i^{-1} \sim \sigma_i^1$ for all generators in the braid group. We will show now that the quotient of the braiding group B_n over this relation is the symmetric group S_n. This is an easy consequence of Theorem 2.12.

Corollary 4.3. *The group $B_n/\sim$ is isomorphic with the symmetric group S_n.*

Proof. Notice that the equivalence relation $\sigma_i^{-1} \sim \sigma_i^1$ states that $\sigma_i^{\pm 2} = 1$ for all generators. Note that this is basically the extra relation that we defined in Theorem 2.12 for S_n. We will notate now σ_i for both over- and undercrossings, since they are equivalent.

We already had a surjective homomorphism M (Definition 2.42) from B_n to S_n. It is easy to see that the homomorphism from $B_n/\sim$ to S_n remains surjective because we can still map generators σ_i in $B_n/\sim$ to generators $(i\ i+1)$ in S_n.

Injectivity now follows immediately since only σ_i and σ_i^{-1} map to $(i\ i+1)$. If this was not the case, we would have, by definition of the initial natural homomorphism M, for an index $j \neq i$, that $(i\ i+1)$ is the same as $(j\ j+1)$, which is obviously not true. Since we had the equivalence $\sigma_i^{-1} \sim \sigma_i^1$, the map is injective.

Thus, the natural homomorphism $B_n/\sim \longrightarrow S_n$ is a bijection, and thus an isomorphism. □

We now can clearly see that B_n/P_n is also isomorphic with $B_n/\sim$. Recall that the crossing types did not matter in B_n/P_n and thus these sets are identitical.

Now clearly, If we do not distinguish between over and undercrossings in braids, we gain no information from braids that we could not get out of a permutation.

Remark 4.4. It would be nice to have an idea of what this equivalence with S_n looks like in the braiding group B_n. We can argue there is no canonical map from S_n to B_n, because B_n is bigger and clearly S_n is not a subset of the set of braids. Recall the two examples of pure braids in Example 2.45. These two braids have the same permutation, since they are both pure braids, and thus a map from S_n to B_n has to choose one pure braid to be the image of the idenitity permutation.

Assume that there is one, denoted $f : S_n \to B_n$. The relation used in Corollary 4.3 induces a canonical isomorphism between S_n and $B_n/\sim$. We can find ways to lift $B_n/\sim$

to B_n. Since, we do not have any specified crossings in $B_n/\sim$, we can flip the signs of the crossings in any lift to B_n to get a totally different map, but equally natural. Thus, there is no natural way to translate the equivalence with S_n to braids. Any map from S_n to B_n is equally valid. There is thus no point in considering specific braids from permutations.

Considering that crossing types do not matter, if we consider a braid with k crossings, we can form 2^k different braids in B_n that have different types of crossings. These different braids are equivalent in $B_n/\sim$, but not in B_n obviously. Hence, we have a lot of different equally valid maps into B_n that differ in crossing types but also in what pure braid is appended to it.

In a lift from B_n/P_n to B_n, an obvious choice for the pure braid is the identity braid, i.e.,

$$Id : B_n/P_n \to B_n : \gamma \mapsto \gamma. \tag{4.10}$$

This is not yet well-defined since crossing types do not matter either in B_n/P_n. Choosing all the crossing homogeneous will result in the so-called positive (or negative) permutation braids. The dissertation [1] defines a subset SB_n of the braiding group B_n that contains all positive (respectively negative) permutation braids.

Definition 4.5. ***Positive permutation braids*** *are all braids that only have positive crossings and all pairs of strands cross each other at most once.*

One can define ***negative permutation braids*** *similarly. We denote the set of these braids as* SB_n.

As expected, these will be very similar to the group S_n, since they form class-representatives for the elements in B_n/P_n. Do note that the notation is different to that of the monoid of singular braids from Definition 6.3 $\mathcal{SB}_n$.

We can propose the following from PhD thesis [1]. For a proof, one can consult the following references [1, 21, 8].

Proposition 4.6. *The group endomorphism from* B_n *to* S_n *restricted to the domain of* SB_n *is bijective with* S_n*, i.e., two braids* $\beta_1, \beta_2 \in SB_n$ *are the same if and only if the induced permutations are the same.*

Note that this is not a group isomorphism, or a well-defined group homomorphism (restriction of the domain to SB_n), since SB_n is a subset and is not closed under the composition braid operator $*$. Clearly, the composition of twice σ_1, σ_1^2, contains two strands that cross each other more than once.

These permutation braids are thus an example of how the set $\mathcal{P}(X, Y)$ can map into B_n.

We now have an idea what the set of voice leadings from a chord $X \in \mathbb{A}_n$ to $Y \in \mathbb{A}_n$ looks like, but this does not allow us to consider differences between chord progressions, e.g., differences between progressions from $\mathcal{P}(X, Y)$ and $\mathcal{P}(Z, Y)$ with $X \neq Z$ such that they differ in at least one note. However, given (3.10), there is a well-defined natural metric on the space $\mathbb{A}_n$ that indicates distances between chords.

We would like to have this difference in chords also visible in the braids. Ideally, we can find a pure braid that embodies the chord distance, such that the appended pure braid does not interfere with the permutation of the braid, and thus the other important piece of information that embodies the voice leading permutation.

Recalling Proposition 3.3, if we equip the natural metric on $\mathbb{A}_n$ with the L^1 norm (such that values are in $\mathbb{Z}$), we can find braids in B_2 that embody the distance between the chords X and Y. However, braids in B_2 with an uneven exponent will not be pure braids as they will have a permutation $(1\ 2)$. We can account for this by making the isomorphism between B_2 and $2\mathbb{Z}$, i.e.,

$$\mathbb{Z} \to B_2 \,:\, z \mapsto \sigma_1^{2z}. \tag{4.11}$$

Recall from (2.30) that we can map any braid from B_2 into B_n. Additionally to this, if we consider braids in B_n, instead of adding these σ_1^{2z} generators, we can use σ_n^{2z}. This has as a result that that part of the braid does not interact with the following braid word representing the permutation from S_n, i.e., in a sense that generators get cancelled which makes it hard to distinguish for example the initial permutation. This is valid since the equivalence between $\mathbb{Z}$ and B_2 is based on the fact that B_2 has one generator. Hence, we can consider σ_n as that sole generator for this isomorphism.

Since the permutation braid is in B_n, adding the extra $(n+1)$th strand will not affect this braid, and since we add a pure braid the permutation also will not be changed. That is, of course considering the $(n+1)$th element is a fixed point in the resulting permutation.

Since the permutation does not change by this pure braid, this will result in the $(n+1)$th strand linking k times with the knot from the permutation braid, with k the chord distance between the two chosen chords from the chosen natural metric.

This way, we can represent chord progressions in B_n by applying the pure braid as defined above, representing the chord distance, and the permutation braid representing the voice leading permutation braid based on some initial permutation representatives and chord orders.

5 | General Ordered Representation

In this chapter, we will have a look at some representations that involve orderings. More specifically we will consider a new representation to strengthen the one from Section 3.1. Namely, a representation that does distinguish between over and undercrossings. We also talk very shortly about ordered chords, and what their progressions entail and, eventually, a general representation.

In Section 5.1, we will look into how the set of braids is restricted by introducing order upon the strands. We will conclude that, while there are many such braids with ordered strands, the number of these braids is still significantly smaller than what could be possible in B_n. Consequently, we can put bounds on the knot genus and the linking number for such braids with ordered strands.

As with Chapter 4, we will algebraically prove an equivalence with the Cartesian product of symmetric groups S_n, and will come up with similar problems, e.g., lack of origin, multiple isomorphisms. In Section 5.3, we solve this issue by introducing additional structure, i.e., we allow for chords to have an order on the notes. This way, we allow for a natural base progression based on the orders of the notes. With this additional structure, we will again try to the voice leadings with ordered changes from $S_n \times S_n$ to B_n, in Section 5.4. However, in Remark 5.35, we will find that the suggested map is not injective when mapping to B_n.

5.1 | Braid representation from Section 3.1

We can apply the idea of ordered voice leadings to actual braids by going back to the representation suggested in Section 3.1. We define over- and undercrossings based on the order of the voices that cross, e.g., for two voices v_1 and v_2 with $o(v_1) < o(v_2)$ (recall Remark 2.32 for o-notation), a crossing between the two strands representing v_1 and v_2 results in the strand of v_1 going over the strand of v_2. This way, the crossings show us the order relation of the strands.

However, requiring such an ordering drastically reduces the types of admissible braids as we will explain below. Intuitively, braid strands never intertwine but always stay on their respective 'layer'. In other words, a strand related to a voice 1 always goes on top of every other strand, then the second voice goes below the first but over all the others, etc. The braid crossings should abide by a strict partial ordering (Definition 2.27) of the voices. This requires a strict transitivity condition, i.e., if we have three strands s_1, s_2, s_3 a with crossings s_1 over s_2, and s_2 over s_3, i.e.,

$$o(s_1) < o(s_2) \text{ and } o(s_2) < o(s_3) \Rightarrow o(s_1) < o(s_3), \tag{5.1}$$

we thus have that strand s_1 also goes over s_3 and cannot cross below strand s_3.

Example 5.1. A braid $\sigma_1\sigma_2^{-1}\sigma_1$ corresponds to voice 1 going over voice 2 (σ_1), voice 1 going under voice 3 (σ_2^{-1}), and voice 2 going over voice 3 (σ_1), which leads to a contradiction since the first two crossings give $o(1) < o(2)$ and $o(1) > o(3)$ and thus by transitivity $o(2) > o(3)$, i.e., voice 3 going over voice 2, but we have voice 2 going over 3.

The example in Example 5.1 shows that not all braids are in accordance with a partial ordering. Note that this is a strict ordering, as strands that 'are equal' collide and create singular points, and this is not valid for a braid in B_n.

From now on, we will mostly refer to braids with a valid order on the strands as valid braids or braids with a valid order.

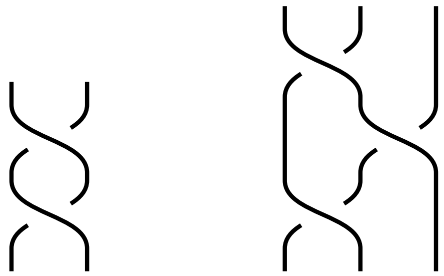


Figure 5.1: Two braid words representing the Hopf link (Figure 2.9), σ_1^2 and $\sigma_1\sigma_2\sigma_1$ respectively, but (a) clearly has no ordering of strands, but (b) does .

Remark 5.2. It is important to note that the set $\{1, \ldots, n\}$, indexing the braid strands, with relations defined by the crossings is not a poset. For example, reflexivity ($x \preceq x$) is not directly deducible from the braid. But, we could argue that the twist Reidemeister move (Figure 2.7) embodies reflexivity, as this twist is the strand going over itself, i.e., $x \preceq x$ or $x \succeq x$. This move is homotopic to just the straight strand, and thus reflexivity is not directly visible in the Artin braid group, but is then captured by homotopy in an equivalent definition[11]. As for transitivity, $\left[x \preceq y,\, y \preceq z\right] \Rightarrow x \preceq z$, the relation $x \preceq z$ might not be literally visible in the braid, but we will assume that this relation is implied by the previous relations. Antisymmetry will simply not occur because that implies that two distinct strands are the same, which is counterintuitive for braids, and otherwise it becomes a contradiction.

One could interpret the set of crossings as defining a 'generator' for a partial order.

Also, if such an ordering exists it is not a knot invariant. In Figure 5.1, we have two braids that both give the Hopf link, but (b) has a valid order on the strands, while (a) has not. This example suggests that one needs bigger braids, with more strands, to consider a certain knot having a valid order on the strands. And clearly, equivalent knots (by Markov equivalence), do not induce the same order. However, we will see that equivalent braids ((2.19)) do have equivalent orders.

Lemma 5.3. *A partial order on the strands is a braid invariant such that equivalent braids ((2.19)) induce the same partial order.*

Proof. Let us denote an index i for a strand based on the starting position, $o(i)$ as the order of the strand, and $p(i)$ as a certain position of strand i in the braid as a consequence of the permutations, e.g., after a braid σ_1 we have that $p(1) = 2$ and $o(1) < o(2)$ (since 1 went over 2, it has a higher position in the ordering[12]). The map p is basically the inverse permutation induced by M of the previous crossings.

We need to show that the partial ordering of strands remains invariant under the braid equivalence relations. For the first relation, $\sigma_i\sigma_j = \sigma_j\sigma_i$ with $|i - j| > 1$, let us consider strands s_1, s_2, s_3, s_4 such that $p(s_1) = i,\ p(s_2) = i + 1,\ p(s_3) = j$, and $p(s_4) = j + 1$. Note that the notation of the strings is arbitrary and that these can be any strings, i.e., as if we consider the changes in the middle of an arbitrary braid.

[11] This definition defines the braid group as $B_n = \pi_1(UConf_n(\mathbb{R}^2))$, i.e., the fundamental group of the nth unordered configuration space of $\mathbb{R}^2$. Notice that it is unordered such that the points are invariant under the symmetric group, and thus different permutations can be obtained with homotopic loops. The ordered variant results in the pure braid group P_n.

[12] We note that the ordering implies that 1 is highest and thus on top

Because $|i-j|>1$, we have that s_1 to s_4 are four different strands as the crossings act on different strands. Thus, the two relations $s_1 R s_2$[13] and $s_3 R s_4$ are independent of each other and will respect an ordering if and only if the initial braid did as well.

For the equivalence of $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, we have three strands s_1, s_2, s_3 such that $p(s_1) = i$, $p(s_2) = i+1$, and $p(s_3) = i+2$. The braid word $\sigma_i\sigma_{i+1}\sigma_i$ now defines the following ordering:

$$o(s_1) < o(s_2) \quad o(s_1) < o(s_3) \quad o(s_2) < o(s_3), \tag{5.2}$$

i.e., strand s_1 on top, then s_2, and s_3 at the bottom. The word $\sigma_{i+1}\sigma_i\sigma_{i+1}$ denotes the following relations:

$$o(s_2) < o(s_3) \quad o(s_1) < o(s_3) \quad o(s_1) < o(s_2), \tag{5.3}$$

which is the same order relation.

Clearly, if we have an occurrence of $\sigma_i\sigma_i^{-1}$, this implies the order $o(s_1) < o(s_2)$ twice, for $p(s_1) = i$ and $p(s_2) = i+1$. And thus this is a valid, but redundant order.

The order on strands thus remains invariant under these equivalence relations. □

We now know that considering orders of the strands is well-defined with the braiding group B_n. And thus, the following statements on the braid word of a braid with a valid order on the strands are well-defined, i.e., all the equivalent braid words will act in the same particular way.

Remark 5.4. The opposite of Lemma 5.3 is not true, i.e., the same order on strands does not imply equivalent braids. Consider braids $\sigma_1\sigma_2^{-1}$ and $\sigma_1\sigma_2^{-1}\sigma_1^{-1}$. These both induce the same order:

$$o(s_1) < o(s_2) \quad o(s_3) < o(s_1) \quad \Big(o(s_3) < o(s_2)\Big), \tag{5.4}$$

with the last σ_1^{-1} generator clearly implying an already implied relation. But these two braids do differ. They have one and two components respectively and are thus not the same braid in B_3. Recall, that the number of components is solely determined by M from Definition 2.42, and that last redundant generator σ_1^{-1} does change the permutation.

Remark 5.5. Note that the idea is here to have a total ordering of the strands, but that total order does not have to be visible in the braid. If a braid is too short to display all the relations, it becomes a partial ordering (Definition 2.27). As long as the partial ordering is a subset of the total ordering, it is appropriate.

Remark 5.6. Note, however, that the representation in Section 3.1 already suggests such ordering since it requires ordering the voice leadings based on their pitches such that the actual voice crossings correspond with crossings of the strands representing the voice leadings. Hence, one could define crossings by choosing the highest voice as the 'top' strand and the lowest voice as the 'bottom' strand. Of course any other order can be considered on the voice leadings, but since we consider this ordering based on the pitches, we already have thus a natural order.

[13]Here, the notation aRb indicates that there is a relation between a and b, but it remains unspecified since actual relation does not matter. Just the fact that there is one matters.

Let us try to consider all braids that abide by such an ordering, not just the natural order as in the above remark, i.e., braids where the strands induce any valid partial order. We want to make a statement on the linking number of the braid strands. But the definition of linking number as in Definition 2.49 between braid strands might be superficial, since two strands might end up being the same link component, e.g., in the braid σ_1 strand 1 and 2 form one component in the closed braid. Hence, let us consider the following alternative definition:

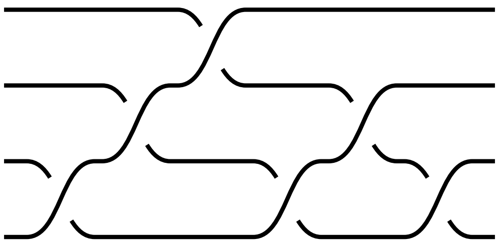
Definition 5.7. *The* ***braid linking number*** *of two strands β^i, β^j is given by:*

$$l_{B_n}(\beta^i, \beta^j) = c_1 + c_2 - c_3 - c_4, \tag{5.5}$$

where c_1, c_2, c_3, c_4 are the number of occurrences of each type of crossing, displayed in Figure 2.10, respectively.

Since a braid word like $\sigma_i \sigma_i^{-1}$ results in a braid linking number of zero for the affected strands, and the braid relations (in (2.19)) do not change the amount or type of crossing, the braid linking number becomes a braid invariant, i.e., equivalent braids will have the same braid linking number between strands. However, this cannot be a valid knot invariant since knots do not have unique braiding representations; Recall Figure 5.1. Recall that the number of components is uniquely defined by the permutation of a braid. In fact, the components can be defined by the disjunct cycles of the permutation, where each cycle contains the indices of the strands that go into the component represented by the cycle.

Example 5.8. The braid $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ gives permutation (1 4)(2 3). The braid looks as follows:



(5.6)

Hence, for a given braid with a certain permutation π and two different, disjunct cycles $\pi_1 = (i\ \pi(i)\ \dots \pi^m(i))$ and $\pi_2 = (j\ \pi(j)\ \dots \pi^n(j))$, such that $\pi = \pi_1\pi_2\pi_3$ for some 'remainder' permutation π_3[14], the linking number for the closed braid of those two components is the following:

$$l(K_{\pi_1}, K_{\pi_2}) = \frac{1}{2}\left(\sum_{k=0}^{m}\sum_{l=0}^{n} l_{B_n}(\beta^{\pi^k(i)}, \beta^{\pi^l(j)})\right). \tag{5.7}$$

This definition makes sense since the strands in a cycle belong to the same link component. Thus, if one wishes to get all the crossings of one components, start with an arbitrary strand in that respective component and see where it crosses strands from the other components. For a strand i in a component π_1, the linking of all the crossings it makes with another component π_2 is $\sum_{l=0}^{n} l_{B_n}(\beta^i, \beta^{\pi_2^l(j)})$. Now this is repeated for all the strands i in one component, resulting in the above equation (5.7).

Example 5.9. The linking number between the two components of $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ with permutation (1 4)(2 3) is

$$\frac{1}{2}\Big(l_{B_n}(1,2) + l_{B_n}(1,3) + l_{B_n}(4,2) + l_{B_n}(4,3)\Big) = -2. \tag{5.8}$$

[14]Note that the order of the disjunct cycles in π does not matter since they are disjunct.

We can prove the following lemma:

Lemma 5.10. *If all pairs of strands* (β^i, β^j) *in a braid abide to a partial ordering, then their braid linking number* $l_{B_n}(\beta^i, \beta^j)$ *has absolute value smaller or equal to one, i.e.,*

$$|l_{B_n}(\beta^i, \beta^j)| \leq 1. \tag{5.9}$$

Proof. We will use crossing types defined in Figure 2.10, and assume w.l.o.g. that i is the red string and j the blue string. The values c_1 to c_4 denote the number of occurrences of each respective type.

Notice that the occurrence of either both type 1 and 2, or both type 3 and 4 will result in a contradiction. The occurrence of non-zero c_1 and non- zero c_2 signifies that $o(i) > o(j)$ and $o(i) < o(j)$ marking a contradiction with the partial ordering. The case of both type 3 and 4 crossings will have an analogous contradiction. It is important to note that there is a constraint on how crossings occur. Obviously, if i has crossed j, in the sense that i is now situated above j in the braid, this cannot occur again. To refer again to the Figure 2.10, a type 3 crossing cannot occur immediately after a type 1 crossing, as the red (ith) string is already 'above' the blue (jth) string. We can notate this condition as follows for variables 'up' u and 'down' d:

$$|u - d| \leq 1 \qquad \text{for} \quad u = c_1 + c_3, \;\; d = c_2 + c_4, \tag{5.10a}$$

$$\text{or equivalently:} \qquad |c_1 + c_3 - c_2 - c_4| \leq 1. \tag{5.10b}$$

Either the ups and downs are equal, $u = d$, implying no change in order w.r.t. to the two strand i and j, or there is a difference of plus or minus. A difference of plus or minus one signifies a change of order and the sign is determined by the initial ordering. This condition is not a condition for the partial ordering, but a more fundamental one for the basic existence of the crossings.

Since we have $c_i \geq 0$, we can rewrite the objective as follows, for a $k \in \mathbb{Z}$:

$$c_1 + c_2 = c_3 + c_4 + k \qquad \text{such that } |k| \leq 1 \tag{5.11}$$

Without loss of generality, we assume that $k > 0$ is strictly positive, and that thus there are more occurrences of type 1 and type 2 crossings than type 3 and type 4. The case for $k = 0$ is of course trivial. The case of $k < 0$ will give a similar result using a contradiction from the occurrence of type 3 and 4 crossings.

As suggested, the extra k crossings of type 1 and 2 must occur either all as a type 1 or all as type 2 crossings, as occurrence of both will result in a contradiction. Again w.l.o.g., we shall assume that all k occurrences are of type 1, and we thus have $c_1 > 0$ because $k > 0$. Then, we can assume that $c_2 = 0$, otherwise a contradiction will occur. Similarly, we can assume that at least one of c_3 and c_4 is zero. We assume w.l.o.g. that $c_4 = 0$. We can rewrite (5.11) as follows:

$$c_1 = k + c_3. \tag{5.12}$$

When we substitute this in the 'ups and downs' (5.10) constraint we get the following:

$$|k + 2c_3| \leq 1. \tag{5.13}$$

With $k > 0$ and $c_3 \geq 0$ by definition, we have $k \leq k + 2c_3$ and thus we can write the following:

$$k \leq k + 2c_3 \leq 1. \tag{5.14}$$

This proves the objective. All the other cases are analogous. □

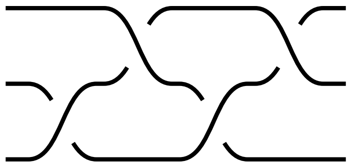


Figure 5.2: A counter example for Lemma 5.10, with braid word $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$.

The reverse implication of Lemma 5.10 is not true, i.e., a braid linking number smaller or equal to one in absolute value for all strands will not per se imply a partial ordering of the strands. An easy counter example is shown in Figure 5.2.

Having bounds on the braid linking number, we can put a bound on the linking number knot invariant using (5.7). We can prove the following:

Theorem 5.11. *For a link coming from a closed braid with a valid partial order on the strands in the braiding group B_N, the linking number between two components from different disjunct cycles $\pi_1 = (i\ \pi(i)\ \ \ldots\pi^m(i))$ and $\pi_2 = (j\ \pi(j)\ \ \ldots\pi^n(j))$ of the braid is bounded in the following way:*

$$|l(K_{\pi_1}, K_{\pi_2})| \leq \left\lfloor \frac{(m+1)(n+1)}{2} \right\rfloor \leq \left\lfloor \frac{N^2}{8} \right\rfloor. \tag{5.15}$$

Proof. Firstly, note that the lengths of the cycles m and n are strictly smaller than N, i.e., $m < N$ and $n < N$, and we have the constraint $n + m + 2 \leq N$[15], such that these cycles actually form disjoint subsets of the N strands. Obviously, the bound will be maximum if all strands are present in the two cycles, i.e., $m + n + 2 = N$.

From the formula in (5.7), we notice that this is maximal if all braid linking numbers $l_{B_n}(\beta^{\pi^k(i)}, \beta^{\pi^l(j)})$ are the same sign and maximally 1 in absolute value, by Lemma 5.10, we get the following:

$$|l(K_{\pi_1}, K_{\pi_2})| \leq \frac{1}{2}\left(\sum_{k=0}^{m}\sum_{l=0}^{n} 1\right) = \frac{1}{2}(m+1)(n+1), \tag{5.16a}$$

$$\leq \frac{1}{2}\left(\frac{N-2}{2}+1\right)\left(\frac{N-2}{2}+1\right) = \frac{N^2}{8}, \tag{5.16b}$$

with the last inequality coming from maximising $(m+1)(n+1)$ under $m + n + 2 = N$[16].

And because the linking number always is a whole number we can add the floor operator to this bound. □

We can now see that for braids in B_3 with ordered strands, the maximally allowed linking number between components is 1.

We continue to characterise those particular braids, with ordered strands. The previous lemma, Lemma 5.10, can tell us the following for the respective braid words.

Lemma 5.12. *Any non-trivial*[17] *occurrence of the braid word $\sigma_i^{\pm 2}$, up to the braid equivalence relations ((2.19)), in a braid will not adhere to a partial ordering.*

[15] Note that the permutations have length $m + 1$ and $n + 1$ respectively. Thus, they embody $m + n + 2$ unique strands.

[16] One can use Lagrange multipliers and notice that the there are sufficient conditions to assume the solution as optimal.

[17] Not a braid like $\sigma_1^2\sigma_1^{-1}$, where the exponent gets reduced to 1.

Proof. Let us again denote an index i for a strand based on the starting position, $o(i)$ as the order of the strand, and $p(i)$ as a certain position of strand i in the braid as a consequence of the permutations.

We shall first prove that the occurrence of σ_i^2 in a braid implies a contradiction with the partial ordering. Thus for σ_i^2, we have for some strands s_1 and s_2 with $p(s_1) = i$ and $p(s_2) = i + 1$, that σ_i interchanges them with $o(s_1) < o(s_2)$, i.e., s_1 goes over s_2. Now we have $p(s_1) = i + 1$ and $p(s_2) = i$, and thus σ_i again implies the initial positions $p(s_1) = i$ and $p(s_2) = i + 1$, but more importantly, $o(s_2) < o(s_1)$, as s_2 now goes over s_2. This is a contradiction with a valid order. The case of σ_i^{-2} is analogous. □

This lemma has the following consequence on pure braids.

Corollary 5.13. *The only pure braid (Definition 2.44) that adheres to a partial ordering is the trivial braid 1.*

Proof. Recall that the generators of the pure braid P_n group were of the form

$$A_{j,k} = \left(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_{j+1}\,\sigma_j^2\,\sigma_{j+1}\cdots\sigma_{k-1}\right), \quad \text{for } 1 \leq j < k \leq n, \tag{5.17}$$

and note the appearance of σ_j^2 in the middle. Now, by Lemma 5.12, the only admissible braid in P_n has all exponents of the generators zero. This is clearly the trivial braid 1. □

The braid from Figure 5.2 is again a counter example of the other implication for Lemma 5.12, i.e., no occurrence of a sub-word σ_i^2 does not imply a valid order. As suggested before, we need more strands to get a braid with a valid order on the strands to get a certain knot. Recall the example in Figure 5.1.

One might intuitively expect that the "most complex" braids are those whose strands are totally ordered, since additional ordering relations generally force more crossings. For example, Lemma 5.10 implies that more relations increase the braid linking number.

Knots themselves cannot be uniquely characterised. It is difficult to consider an arbitrary knot and conclude it is actually the trefoil knot. This is the whole idea of knot theory. There is no definition that can be checked to conclude the identity of a knot by using, for example, a knot table [18] For example, a knot can have many different braid representations, and thus a knot cannot be represented uniquely by a braid word. The only thing that remains unique or invariant for knots are knot invariants.

Similarly to the 'definition of a knot', we can also not uniquely consider complexity of knots, because we do not have a grasp of what exactly this knot is. Again, the only thing that can be considered is that which is fixed for knots, knot invariants. Thus, we can only consider complexity regarding a knot invariant. Perhaps we can conclude bounds on it, as with Theorem 5.11, or make an ordering of knots based on said knot invariant. And that is as far as we can take it considering 'complexity'.

Thus, our statement on braids that exhibit a total order on the strands being most complex is not a well-defined one. We can, however, conclude that for example with the linking number Lemma 5.10, there is a bound on the amount of linking, thus a bound on the amount of meaningful crossings. Braids having a total order require the most number of relations in general. So we can say it will generally need more crossings to display such a total order, making it more likely that such a braid will hit the proven bounds on the linking number. And thus, considering this knot invariant, one could argue that such braids are indeed more complex w.r.t. the linking number.

[18] See appendix of [3] for a knot table.

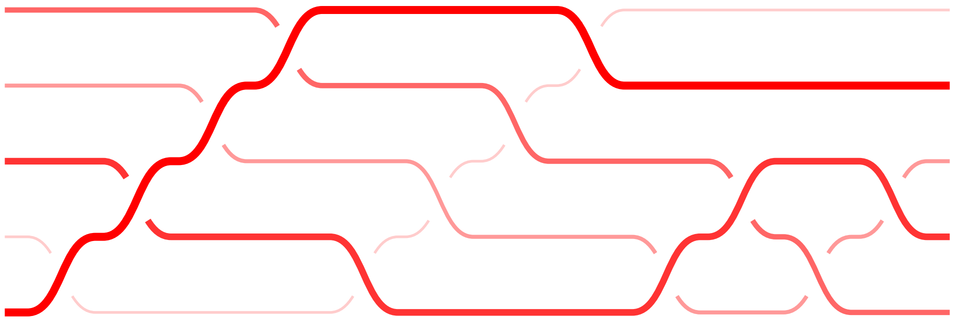


Figure 5.3: A braid $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$ that has partially ordered strands. This braid contains both positive and negative crossings. The tints of red and thickness denote the ordering, with the brightest and biggest strand first

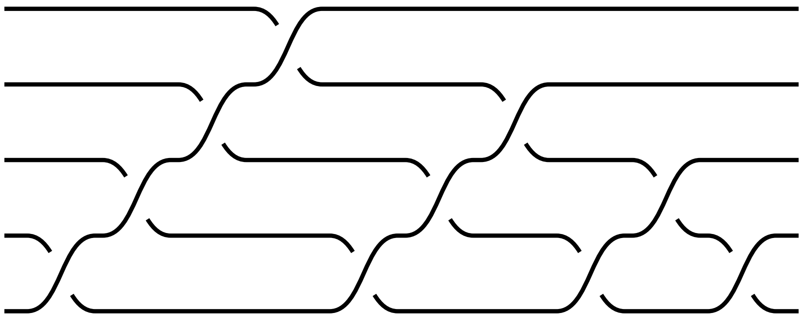


Figure 5.4: The fundamental braid Δ_5 from Definition 5.14.

Knowing that not every braid admits a valid order on the strands, it might be useful to consider braids that have a total order or attain the maximal bound for a knot invariant to consider the limits of braids with a valid order.

To get back to such braids, recall the permutation braids from Definition 4.5. We will show in Theorem 5.17 that these admit a valid order on the braid strands.

But not all braids that have partially ordered strands are positive permutation braids. Take for example the braid $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}$, as shown in Figure 5.3, which has both positive and negative crossings.

It has been proven [1] that a braid β is a positive permutation braid, in SB_n, if and only if the braid is a factor of the fundamental braid Δ_n. The latter fundamental braid has the following definition:

Definition 5.14. *The* ***fundamental braid*** Δ_n *in* B_n *is defined as follows:*

$$\Delta_n = (\sigma_1 \ldots, \sigma_{n-1})(\sigma_1 \ldots, \sigma_{n-2}) \ldots (\sigma_1\sigma_2)(\sigma_1) = \Pi_{n-1}\Pi_{n-2} \ldots \Pi_2\Pi_1, \tag{5.18}$$

with

$$\Pi_i = \sigma_1\sigma_2 \ldots \sigma_{i-1}\sigma_i. \tag{5.19}$$

An example of Δ_5 is shown in Figure 5.4. One clearly has that the induced permutation from M (Definition 2.42) of Δ_n is the following

$$M(\Delta_n) = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ n & n-1 & \ldots & 2 & 1 \end{pmatrix}. \tag{5.20}$$

Note that this is not the cycle notation but the two-line notation. Clearly, this is also the permutation $M(\Delta_n^{-1})$ of the inverse of the fundamental braid.

As we mentioned before about factors of the fundamental braid, we have the following theorem from [1]:

Theorem 5.15. *A braid β is in SB_n (Definition 4.5) if and only if there exists a positive permutation braid $\gamma \in SB_n$ such that $\Delta_n = \beta\gamma$ or $\Delta_n = \gamma\beta$, with Δ_n the fundamental braid.*

Proof. We shall provide a conceptual proof. A complete proof can be found in [1].

We consider the left implication. Since the braid γ only has positive crossings and is minimal w.r.t. the permutation, i.e., strands cross each other only once, it will contain no unneeded crossings. This has as a consequence that also the braid β is this way, i.e., it will have no negative crossings as it does not need to undo any crossings from γ, and otherwise the fundamental braid Δ_n will end up with a negative crossing, which is of course not possible. Furthermore, β will have no unneeded changes in its permutation since γ cannot and will not undo it, or otherwise it would be trivial, e.g., like $\sigma_i\sigma_i^{-1}$. This makes that β will just display the permutation that is required to get the permutation of Δ_n, and thus itself will be in SB_n.

As for the right implication, a braid $\gamma \in SB_n$ can be constructed by constructing the valid permutation such that $M(\beta)M(\gamma) = M(\Delta_n)$ or $M(\gamma)M(\beta) = M(\Delta_n)$ (recall Definition 2.42), i.e., $M(\gamma) = (M(\beta))^{-1} M(\Delta_n)$ or $M(\gamma) = M(\Delta_n)(M(\beta))^{-1}$. We can thus find such braid $\gamma \in SB_n$ such that $M(\beta)M(\gamma) = M(\Delta_n)$. It is left to show that we also have $\beta\gamma = \Delta_n$.

We get that in the concatenation of β and γ, there is an even number of extra crossings between two strands compared to Δ_n. This happens because every extra change in permutation also needs to be undone again to remain the same permutation of Δ_n. With both γ and β being positive permutation braids, they have at most one crossing between strands. Thus, in the concatenation the number of crossings between two strands is either 0, 1, or 2. Since both γ and β have positive crossings, they cannot be cancelled in the concatenation by an inverse relation. And two strands in the fundamental braid also cross each other at most once by definition (see Figure 5.4). Thus, with the even extra number of crossings, we have in the concatenation an odd amount of crossings between strands. Thus, it has to be 1 crossing between strands in the concatenation. We conclude that with the permutation the same, and the same amount of crossings we have that $\beta\gamma = \Delta_n$. □

Here a positive braid is one with only positive crossings (see Remark 2.48). As a result of this, every permutation $\pi \in S_n$ induces a positive permutation braid β. And a positive permutation braid also induces another positive permutation braid β^* by above Theorem 5.15. And a positive permutation braid is equivalent with a permutation by Proposition 4.6. Thus, there is another respective permutation $\pi^* \in S_n$, such that $\Delta_n = \beta\beta^*$ and $M(\Delta_n) = \pi\pi^*$, or $\Delta_n = \beta^*\beta$ and $M(\Delta_n) = \pi^*\pi$.

We have the following result for permutation braids:

Lemma 5.16. *The fundamental braid Δ_n induces a valid order on the strands*

Proof. We show by induction that Δ_n has a valid ordering of the strands. For $n = 2$, we have $\Delta_2 = \sigma_1$ which orders the two strands. Assume now that Δ_k orders k strands. Note that we have $\Delta_{k+1} = \Pi_k\Delta_k$. Also, the induced permutation of Π_k, $M(\Pi_k)$, is the following:

$$M(\Pi_k) = \begin{pmatrix} 1 & 2 & \dots & k & k+1 \\ 2 & 3 & \dots & k+1 & 1 \end{pmatrix}. \tag{5.21}$$

Thus, since by assumption Δ_k orders the strands, we know by the permutation $M(\Pi_k)$ in $\Delta_{k+1} = \Pi_k\Delta_k$, the braid Δ_k orders the strands indexed by $2, 3, \dots, k, k+1$. So what is left to show is that the strand indexed 1 will be added to this chain in a way that does not induce a contradiction.

Indeed, the braid Π_k in the beginning assumes the following relations:

$$o(s_1) < o(s_i), \qquad \forall i \in \{2, 3 \ldots, k, k+1\}, \tag{5.22}$$

with s_i the strands indexed as i. This makes the strand s_1 all the way at the top of the braid, or at the bottom of the chain (w.r.t. the order $o(s_i)$).

Thus, by induction, Δ_{k+1} also has ordered strands. □

We can use this to now prove the general result for the positive permutation braids.

Theorem 5.17. *All positive (or negative) permutation braids $\beta \in SB_n$(Definition 4.5) adhere to a partial ordering of the strands.*

Proof. The proof will be analogous for negative braids, and thus we shall continue the proof for the positive case.

By Theorem 5.15, we know that all positive permutation braids are sub-words, up to braid equivalence ((2.19)), of the fundamental braid. By Lemma 5.16, we know that the fundamental braid adheres to a partial ordering.

Since the given permutation braid β and the other factor γ of the fundamental braid (by Theorem 5.15) only use one type of crossing (negative or positive), there will be no cancellation of certain crossings by concatenating the two. With the number of crossings not changing by braid relations, we get that the sum of the number the crossings of β and γ will be exactly the length of the fundamental braid, i.e., $|\Delta_n| = |\beta| + |\gamma|$ with $|\beta|$ denoting the word length. We also know that by braid relations the order on the strands does not change (Lemma 5.3), such that the two factors each embody a part of the order relations. That is, w.l.o.g., if we have $\Delta_n = \gamma\beta$, the crossings in Δ_n can be partitioned such that β and γ each have their disjunct set of relations. With β behind γ in the concatenation, β will define its relations of Δ_n by asserting its own order on the permutation of γ, $M(\gamma)$. With Δ_n having a valid order, the relations of β will be exactly a subset of those in Δ_n up to the permutation $M(\gamma)$. Thus, just considering β, it will have a valid order. If we have $M = \beta\gamma$, we can make the same argument but now β will have the exact same relations as displayed in the fundamental braid Δ_n. This happens because we do not need to account for the permutation of γ, that came before β earlier. □

Remark 5.18.

- Notice that the argument in Theorem 5.17 does not work the other way around, and that is thus also not the argument we are making. That is, concatenating two braids with valid orders will not per se result in a braid with a valid order. The easiest example of this is concatenating σ_1 with itself to get σ_1^2.
- Also, the idea of a being sub-word (factor) of a braid with a valid order, having a valid order itself is also not always the case. When we consider mixed (w.r.t. the sign) crossings, we can easily form a contradiction:

$$\beta_1 = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}, \qquad \beta_2 = \sigma_2\sigma_1^{-1}, \qquad \beta = \beta_1\beta_2 = \sigma_1\sigma_2^{-1}. \tag{5.23}$$

  Here, β_1 is the counterexample from Figure 5.2 (with thus no valid order), and β has a valid order.

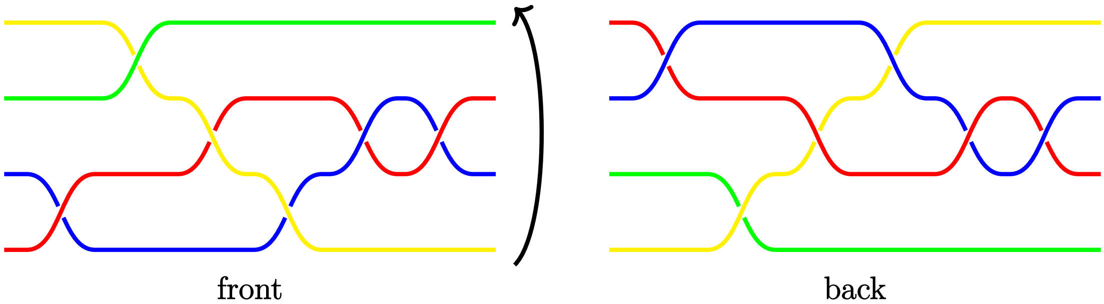


Figure 5.5: Front and back view of a braid, with front braid word $\sigma_1\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2^2$ and back braid word $\sigma_3\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_2^2$. Note that these are each other's image of the Flip map.

We now know that the positive or negative permutation braids are valid braids. We have already argued (with the braid in Figure 5.3) that the set of all braids with partially ordered strands will be more complex than SB_n. Let us define the following operations on the braids in order to create more braid words with valid orders on the strands.

Definition 5.19. *The* ***mirror braid*** Mir : $B_n \to B_n$ *maps all overcrossings to undercrossings and vice versa, i.e.,*

$$\text{Mir}(\sigma_i^{\pm 1}) = \sigma_i^{\mp 1}, \qquad \forall 1 \leq i < n. \tag{5.24}$$

The map Mir is clearly a group homomorphism.

Definition 5.20. *The* ***flipped braid*** Flip : $B_n \to B_n$ *map is a right conjugation map of* Δ_n *, i.e.,*

$$\text{Flip}(\beta) = \Delta_n^{-1}\beta\Delta_n. \tag{5.25}$$

One can show [8, Thm 5.11] the following for generators σ_i

$$\text{Flip}(\sigma_i) = \sigma_{n-i}, \qquad \forall 1 \leq i \leq n-1. \tag{5.26}$$

The map Flip is a group automorphism. Since, this is a homomorphism, and we have $\text{Flip}(\sigma_i^{-1}\sigma_i) = 1$ we can also conclude that

$$\text{Flip}(\sigma_i^{-1}) = \sigma_{n-i}^{-1}, \qquad \forall 1 \leq i \leq n-1. \tag{5.27}$$

This mapping changes braids as if you were looking at it from behind instead of the front, as is visible in Figure 5.5. Note that if we wish to consider the strands of this back-view braid w.r.t. the strands of the initial braid, we can make the following indexing map:

$$i_{\text{Flip}} : \{1, \ldots, n\} \to \{1, \ldots, n\} : i \mapsto n - i + 1, \tag{5.28}$$

where we map every index i in the initial braid to index $n - i$ in the flipped braid. Note that this is exactly the permutation $M(\Delta_n)$ from (5.20).

Definition 5.21. *The* ***reverse braid*** Rev : $B_n \to B_n$ *maps all braids to their reversed braid words, i.e., for* $\beta = \sigma_{i_1}^{e_1} \ldots \sigma_{i_k}^{e_k} \in B_n$ *we have*

$$\text{Rev}(\beta) = \sigma_{i_k}^{e_k} \ldots \sigma_{i_1}^{e_1}. \tag{5.29}$$

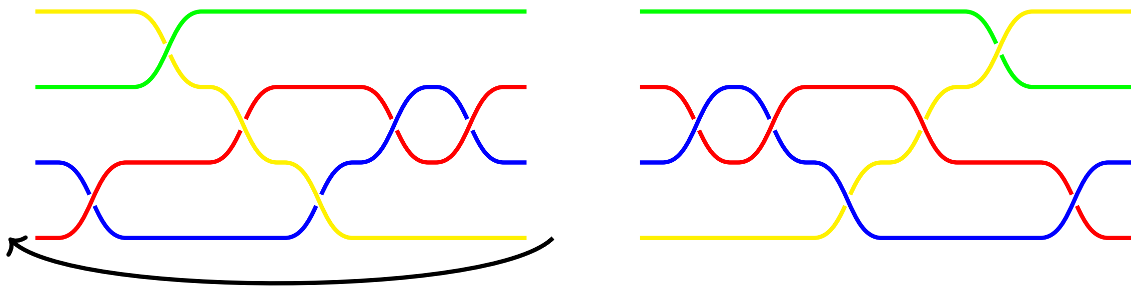


Figure 5.6: Front and back view of a braid, with front braid word $\sigma_1\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_2^2$ and back braid word $\sigma_2^2\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1$. Note that these are each other's image of the Rev map.

The map Rev is clearly not a group homomorphism as the operator also needs to be reversed for it to work, i.e., we have $\mathrm{Rev}(\beta_1\beta_2) = \mathrm{Rev}(\beta_2)\mathrm{Rev}(\beta_1)$. The reversing of a braid is visualised in Figure 5.6.

From Theorem D.1 in Appendix D, we see that the Mir, Rev, and Flip braids also adhere to a partial order of the strands, given an initial braid with a valid order. As a consequence, we also have that the braid inverse has a valid order, as in Corollary D.2.

Note that from the proof of Theorem D.1, we have that the $\mathrm{Mir}(\beta)$ braid induces the exact dual order, but the other maps, Flip and Rev, only give the dual order when changing the indexing of the strands, and thus do not induce the actual dual order. Thus, the inverse braid will not give the dual or same order as the initial braid.

Example 5.22. The braid $\sigma_1\sigma_2^{-1}$ and its inverse $\sigma_2\sigma_1^{-1}$ induce the following orders respectively:

$$o(s_3) \leq o(s_1) \leq o(s_2), \qquad o(s_2) \leq o(s_3) \leq o(s_1), \tag{5.30}$$

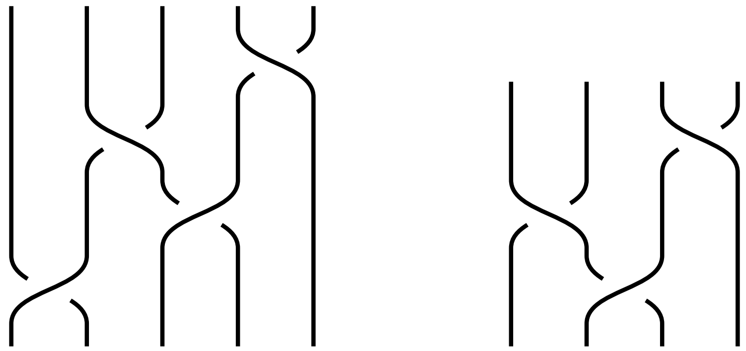
with $s_{1,2,3}$ being the first, second, and third strand respectively.

Clearly any braid in B_n of the form $\sigma_i\sigma_{i-1}\dots\sigma_{k+1}\sigma_k$, with $1 \leq k \leq i < n$, with descending crossings, is also a valid braid. This braid can be constructed by using the Flip operator and using braids of the ascending form $\sigma_i\sigma_{i+1}\dots\sigma_{k-1}\sigma_k$ with $1 \leq i < k < n$ as sub-words from the fundamental braid Δ_n and Theorem 5.17.

These results give an idea of what kind of braids we can construct that adhere to a partial ordering. We know that positive permutation braids have partially ordered strands by Theorem 5.17. Similar to Definition 4.5, one could define the negative permutation braids, which are essentially the same braids but the crossings change from under- to overcrossings. This is exactly the Mir operator on the positive permutation braids. And thus this operation does not construct new braids outside these sets. Similarly, the two other braid operations (flipped and reverse) map positive permutation braids to positive permutation braids. To see this, recall from Definition 4.5 that strands cross only once, and if in the image braid, the strands cross more than once, they are not partially ordered by Lemma 5.10, but this is a contradiction with Theorem D.1. Thus, the reverse and flipped braid also have that strands cross each other just once. Also, the input braid is a permutation braid and the maps Rev and Flip do not change the signage of the crossings. Thus, SB_n is closed under the maps Mir, Flip, and Rev.

An example of a braid that has partially ordered strands but is not a permutation braid was already shown in Figure 5.3.

Let us now consider a small subset of braids with partially ordered strands, namely those that are totally ordered in a minimal amount of crossings. For a braid element of



(a) (b)

Figure 5.7: Totally ordered strands with a minimal amount of crossing; (a) $\sigma_4\sigma_2\sigma_3^{-1}\sigma_1^{-1}$ with four crossings in B_5 and (b) $\sigma_3\sigma_1\sigma_2^{-1}$ with three crossings in B_4

B_n, the minimum crossings required to get a totally ordered set is $n-1$. Connecting all n elements in one chain can be done by $n-1$ links between the n elements. Examples of such braids are shown in Figure 5.7. These minium braids need to have the following structure to be totally ordered:

Lemma 5.23. *A braid in B_n that has totally ordered strands in a minimum of $n-1$ crossings has the following braid word constraints, up to equivalence of (2.19):*

- *No occurrence of $\sigma_i^{\pm 1}\sigma_{i+1}^{\mp 1}\sigma_{i+2}^{\pm 1}$ is possible for any $1 \leq i < n-3$.*
- *Each generator $\sigma_i^{\pm 1}$ appears exactly once for $1 \leq i < n$,*
- *and for all $1 \leq i < n-1$, we have that the exponents of generator σ_i and σ_{i+1} have different sign, i.e., we have either an occurrence of σ_i and σ_{i+1}^{-1} or vice versa.*

Note that the first constraint already incorporates the third constraint in itself. So clearly, any three ascending, adjacent crossings will result in an invalid braid, w.r.t. having a minimal amount of crossings and being totally ordered.

Proof. We consider the first constraint. Using four strands $s_1,\ s_2,\ s_3,\ s_4$ with $p(s_1) = i,\ p(s_2) = i+1,\ p(s_3) = i+2,\ p(s_4) = i+3$, we can show that the braid word $\sigma_i\sigma_{i+1}^{-1}\sigma_{i+2}$ (respectively for the opposite exponents) does not totally order the four strands and thus also not all the strands, i.e., since it is not the most efficient for these four strands, it will not be efficient for all the strands. The implied orders are $o(s_1) < o(s_2)$, $o(s_1) > o(s_3)$, and $o(s_1) < o(s_4)$, which implies no order between s_2 and s_4. It is thus not efficient, since we can easily imagine a braid word as in Figure 5.7 that does order the four strands totally, with fewer crossings.

We consider the second constraint. From Lemma 5.12, we already know that σ_i^2 will not result in an ordering of strands. So if a generator appears twice, we need just one occurrence of a generator $\sigma_{i\pm1}^{-1}$ in between such that the braid word cannot permute to σ_i^2. Two occurrences of $\sigma_{i\pm1}^{-1}$ will lead to the same case as above. An occurrence of both σ_{i+1}^{-1} and σ_{i-1}^{-1} in between will result in a contradiction. Consider $\sigma_i\sigma_{i+1}^{-1}\sigma_{i-1}^{-1}\sigma_i$[19], given four strands $s_1,\ s_2,\ s_3,\ s_4$ with $p(s_1) = i-1,\ p(s_2) = i,\ p(s_3) = i+1,\ p(s_4) = i+2$,

[19]Note that we already incorporate the third constraint. Evidently other cases are not allowed, as we will see with the proof of the third constraint.

the implied orders are $o(s_2) < o(s_3)$, $o(s_2) > o(s_4)$, $o(s_1) > o(s_3)$ and $o(s_1) < o(s_4)$. The latter is in clear contradiction with the order implied by transitivity, $o(s_1) > o(s_4)$. So without loss of generality, let us just consider $\sigma_i\sigma_{i\pm1}^{-1}\sigma_i$. Again, consider strands $s_1,\ s_2,\ s_3$ with $p(s_1) = i,\ p(s_2) = i+1$, and $p(s_3) = i+2$ or $p(s_3) = i-1$, depending on the $i \pm 1$. The braid word $\sigma_i\sigma_{i\pm1}^{-1}\sigma_i$ then gives: $o(s_1) < o(s_2)$, $o(s_1) > o(s_3)$, and $o(s_2) < o(s_3)$ for $i+1$, and $o(s_1) < o(s_2)$, $o(s_3) > o(s_2)$, and $o(s_3) < o(s_1)$ for $i-1$. Both clearly result in a contradictory ordering. Thus, there is no more than one occurrence of the same generator.

If a generator σ_i, for $i = 1, n-1$, has no occurrence then the strand 1 or n is clearly not involved in the ordering and thus this is not a total ordering of strands. If an 'inner' generator σ_i, for $1 < i < n-1$, is missing, we have no interaction between strands at position i and $i+1$ ever in the braid. This then clearly results in two separate order chains, and thus again no total ordering.

We consider the third constraint. If we want σ_i and σ_{i+1} to not be next to each other in the braid word we need a generator in between that is not the same (otherwise have two occurrences), but also not too far off in index such that they can permute next to each other by braid equivalence relations. We need a generator σ_k, in $\sigma_i\sigma_k^{\pm1}\sigma_{i+1}$, such that $|i-k| = 1$ and $|i+1-k| = 1$, but these conditions cannot be met at the same time. We thus will always have that the generators σ_i and σ_{i+1} are neighbours in the braid word.

Note that if we have $\sigma_i\sigma_{i+1}$ for some generator i, that this implies for strands $s_1,\ s_2,\ s_3$ with $p(s_1) = i,\ p(s_2) = i+1,\ p(s_3) = i+2$, we get $o(s_1) < o(s_2)$ and $o(s_1) < o(s_3)$, which is not a total ordering for those three strands as the relation between s_2 and s_3 is not determined. Thus, these two crossings will not result in a minimum amount of crossings for the total ordering of the strands. We can show for $\sigma_i^{-1}\sigma_{i+1}^{-1}$ similarly that it is not efficient since it gives the dual order, using the Mir map from Definition 5.19.

Similarly, by using the map Rev from Definition 5.21, we know that the other occurrences, e.g., $\sigma_{i+1}\sigma_i$, are also not most efficient and do not give a total order. We can make this argument since those maps were shown to give the same or dual order up to a reindexing, shown in Theorem D.1. Thus, if one indexing does not give a total order, neither will the dual nor one with other indices. Hence, the only possible occurrence is one where the crossings types differ for neighbouring indices. It is now not difficult to see that this gives in fact a total order on the three relevant strands. Consider $\sigma_i\sigma_{i+1}^{-1}$ with strands $s_1,\ s_2,\ s_3$, with $p(s_1) = i,\ p(s_2) = i+1,\ p(s_3) = i+2$, we get $o(s_1) < o(s_2)$ and $o(s_1) > o(s_3)$. This is a total order on the three strands. The other cases come from using the maps Mir, Flip, and Rev. This ends the proof. □

Now that we know what such minimal braids with total order look like, we can come to the following conclusion on the knots that form from closing the respective braids.

Theorem 5.24. *All knots resulting from closing braids in B_n with totally ordered strands, with a minimum amount of crossings, are isotopic to an unknot.*

Proof. Consider such a braid $\beta \in B_n$. We will show this using the Markov moves (see Chapter 2). Recall from Lemma 5.23 that all generators appear only once. With Markov conjugation, we will separate the highest indexed generator ($n-1$, then $n-2$, $n-3$,...) to the outside of the braid word. Using Markov stabilisation we can then remove this generator, as the resulting braid will result in an isotopic knot.

Let us show this process for the generator σ_{n-1}. We have the following:

$$\exists \beta_1, \beta_2 \in B_n : \ \beta = \beta_1\sigma_{n-1}\beta_2. \tag{5.31}$$

Using Markov conjugation we get that this braid will be knot-isotopic to $\beta_2\beta_1\sigma_{n-1}$. Using stabilisation, we remove σ_{n-1} to get the isotopic braid word $\beta_1\beta_2 \in B_{n-1}$ that now has length $n-2$ (from the initial length of $n-1$).

We can repeat this process to eliminate all generators to get that the braid β is knot-isotopic to 1. Hence, such a braid β is isotopic to an unknot. □

The result of Theorem 5.24 shows that considering knot invariants will not be very useful, as the braids with totally ordered strands in a minimal amount of crossings are just the trivial unknot.

We can try to consider other braids, but this will get increasingly hard. As we step away from considering optimal crossings to get a total order on the strands, we will get more redundant crossings or not very optimal chain-building. The order of crossings is very important as it shows what is already known in the strand ordering, and what is not. This can make for situations where indeed the next crossing is not as useful in determining the order on the strands. Note that with the permutation of the strands changing after each crossing, we are also very limited in what order relations can come next. And this permutation is again dependent on the previous crossings. So we are not only dependent on the previous crossings for the next relation in the order, but also whether this relation is useful or not.

Example 5.25. An example of a case where the braid is not efficient in its relations is the fundamental braid Δ_n. Here, the first $n-1$ crossings create $n-1$ different chains, i.e., for strands $s_{1,2,3,4,5}$ we get:

$$o(s_1) \leq o(s_2), \qquad o(s_1) \leq o(s_3), \qquad o(s_1) \leq o(s_4), \qquad o(s_1) \leq o(s_5), \tag{5.32}$$

with o again the order of the strands. The next crossings will add relations to this first chain $o(s_1) \leq o(s_2)$, but again create multiple chains out of it, like the following:

$$o(s_1) \leq o(s_2) \leq o(s_3), \qquad o(s_1) \leq o(s_2) \leq o(s_4), \qquad o(s_1) \leq o(s_2) \leq o(s_5). \tag{5.33}$$

This continues until a total order is acquired. Clearly, this is not very efficient. This example is one where there are a maximal amount of relations given without one becoming redundant, i.e., a relation is implied by transitivity of previous ones, or that relation is given again.

Example 5.26. Another example with a redundant crossing was already shown in Figure 5.3. Specifically, the crossing σ_4^{-1} in the braid word $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}$ tells us that strand one goes over the strand two, but this was already given by the first crossing.

Removing this redundant crossing results in a different braid. The initial braid, as shown in Figure 5.3, is a link consisting of one component. Without the crossing σ_4^{-1} we get a link with two components.

While the given relation from this crossing might have been redundant, it has a non-trivial effect on the permutation. In fact, any added crossing will always change the permutation and thus redundancy in the strand order will give non-redundant changes in the braids properties.

This again shows that we can introduce arbitrary complexity in the braid without changing the order on the strands.

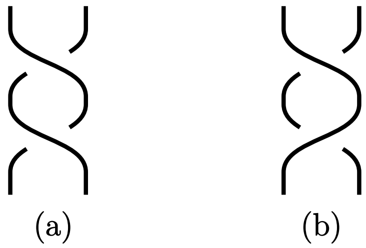


Figure 5.8: Two braids representing the idea of redundant crossings in Definition 5.27. The braid (a) σ_1^2 shows a contradiction of order. The braid (b) $\sigma_1\sigma_1^{-1}$ displays redundant information since the order relation is deduced from both crossings.

While the permutation given from M (Definition 2.42) is a braid invariant, perhaps the knots resulting from closing the braids with and without its redundant crossings could be isotopic, since a braid invariant does not have to be a knot invariant. However, the above example, Example 5.26, is also an example of where this is clearly not the case, as the number of components differ.

And thus this problem is not an easy one to consider. Acquiring a set of all braids with ordered strands is a hard problem. Putting bounds on knot and braid invariants is thus hard since the braid can change in much more arbitrary ways with redundant crossings. Thus, let us just consider these braids that have no redundancy. We define those redundant (and contradicting) crossings as follows:

Definition 5.27. *In a braid word $\beta_1\beta_2\sigma_i^{\pm1}$, that partially orders the strands in B_n, the crossing $\sigma_i^{\pm1}$ is either* ***redundant*** *or a* ***contradiction*** *for two strands s_1 and s_2 if there exists an index $j < n$ such that $M(\beta_1)(j) = s_1$, $M(\beta_1)(j+1) = s_2$, and $M(\beta_1\beta_2)(i+1) = s_1$ and $M(\beta_1\beta_2)(i) = s_2$, and the sub-word β_2 starts with a crossing $\sigma_j^{\pm1}$.*

Recall the list of symbols Table 0.1, where we define s_i to be the index of the strand independent of the braid permutation. The Figure 5.8 displays both cases of redundant information and contradicting crossings. Here, both braids have $\beta_1 = 1$, $\beta_2 = \sigma_1$, and $i = j = 1$.

This essentially means that strands s_1 and s_2 actually do not cross more than once, and not just in the sense of Definition 5.7 and Lemma 5.10, where other trivial occurrences, e.g., $\sigma_1\sigma_1^{-1}$, 'does not cross' as they give linking number zero, but clearly cross twice.

This definition is equivalent with the following statement:

Proposition 5.28. *For a braid word $\beta_1\beta_2\sigma_i^{\pm1}$, Definition 5.27 is equivalent with the existence of $j < n$ such that we have:*

$$j = M(\beta_2)(i+1), \qquad j+1 = M(\beta_2)(i), \tag{5.34}$$

and the sub-word β_2 starts with a crossing $\sigma_j^{\pm1}$

Proof. We have $M(\beta_1)(j) = s_1$, $M(\beta_1\beta_2)(i+1) = s_1$, $M(\beta_1)(j+1) = s_2$, and $M(\beta_1\beta_2)(i) = s_2$ from Definition 5.27. By substituting these into each other, we get the following relations:

$$M(\beta_1)(j) = M(\beta_1\beta_2)(i+1), \qquad M(\beta_1)(j+1) = M(\beta_1\beta_2)(i). \tag{5.35}$$

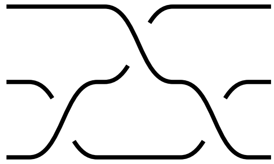
Since M is a homomorphism, we can add the inverse of $M(\beta_1)$ to each side to get the following:

$$j = M(\beta_2)(i+1), \qquad j+1 = M(\beta_2)(i). \tag{5.36}$$

□

This proposition allows an alternative definition that is perhaps more useful for actually checking such redundancies.

While Definition 5.27 does limit explicit redundancy by not allowing strands to cross more than once, redundancy coming from transitivity is still possible. Consider for example the braid $\sigma_1\sigma_2^{-1}\sigma_1^{-1}$, displayed below:



(5.37)

Here, the first two crossings already imply a total order, and the third crossings implies an order relation that is thus already implied by transitivity. However, by Definition 5.27, this has no redundancy.

For braids with no redundancies or contradictions (w.r.t. the order), as in Definition 5.27, we can prove the following:

Theorem 5.29. *Braids with partially ordered strands and no redundancy as defined in Definition 5.27, can be obtained by lifting braids from the quotient B_n/P_n (or equivalently $B_n/\sim$, as in Corollary 4.3) to B_n.*

Proof. Recall the definition of positive (or negative) permutation braids as in Definition 4.5. In those braids strands cross each other just once. Now notice that we have exactly this for the braids with no redundancy. Strands with no redundancy, crossing each other more than once is in contradiction with the definition.

This means that a braid $\beta \in B_n$ with valid partial ordering on the strands and no redundancy is only different from a positive permutation braid $\gamma \in SB_n$ in certain crossing signs, i.e., making all crossings positive in our initial braid gives a positive permutation braid.

But in quotient group B_n/P_n (or $B_n/\sim$), we had that the sign of the crossings did not matter, and so the initial braid and such a positive permutation braid are the same in the quotient group, i.e. $\beta = \gamma$ in B_n/P_n. Also, we have already seen that these positive permutation braids are bijective with S_n (Proposition 4.6), and thus also B_n/P_n. The permutation braids are a specific example of lift from S_n to B_n. Let us notate $l : S_n \to B_n$. Hence, one can find a lift l' that maps a braid in B_n/P_n to that requested braid β by essentially changing the definition of l to account for the valid crossing signs of β. $\square$

We have thus shown that all braids that have an order on the strands can be formed from a map $S_n \to B_n$, such that the composition with M, from Definition 2.42, is the identity map.

Recall that one of the largest non-redundant braids with a valid order on the strands is the fundamental braid. This has as word length $n(n-1)/2$, which is thus the largest amount of crossings we can get without redundancy. Using this, we can put a bound on the knot genus.

Theorem 5.30. *For a knot resulting from closing a braid in B_n with valid order on the strands and no redundancy (by Definition 5.27), we can bound the knot genus g as follows:*

$$g \leq \left\lfloor \frac{n^2 - 3n + 2}{4} \right\rfloor \tag{5.38}$$

Proof. As we mentioned before, $n(n-1)/2$ will be the largest amount of crossings we can get without redundancies. We can use this in the formula (2.36) for the genus of a Seifert surface. Since the actual knot genus, the invariant, is the minimum, this will be a valid bound. The number of components will always be at least one. The substitution in (2.36) will thus give the following:

$$g \leq g_s = \frac{2 + \frac{n(n-1)}{2} - n - 1}{2}, \tag{5.39a}$$
$$\leq \frac{4 + n(n-1) - 2n - 2}{4}, \tag{5.39b}$$
$$\leq \frac{n^2 - 3n + 2}{4}. \tag{5.39c}$$

Since the genus is a whole number[20], we can again take the floor operator to conclude the bound. □

Recall our statement after Theorem 5.11, on valid braids in B_3, the braid group of three strands. From this theorem, we know that the linking number was limited to 1. Now, we know that the genus will be bounded by the floor of 1/2, and is thus always zero, that is for non-redundant braids. And indeed, the only knots that abide by this knot genus and linking number that come from a braid with a valid are the unknot and the Hopf link. This is quite a smaller set of viable knots compared to what is possible in B_3.

5.2 | General Ordered Voice Leadings

Recall, that our finding from Section 3.1 is that a lack of distinction between over and undercrossings results in a lack of authentic complexity in the braiding group. The idea is to have a condition for over and under crossing that is natural and has meaning. As suggested before, we could order each voice leading, and thus also each strand. Musically, this can be interpreted as arpeggiated chords, where notes follow in sequence rather than all at once. We can again consider the set of chord progressions. For non-singular chords $X, Y \in \mathbb{A}_n$ (see Remark 3.6), we denote the set $\mathcal{P}_o(X, Y)$ (as similarly defined in Definition 4.1) as follows:

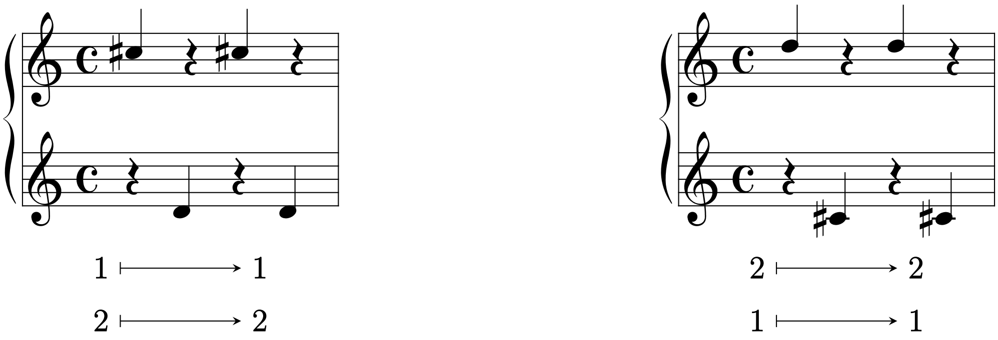
$$\mathcal{P}_o(X, Y) = \Big\{(p, o) \,|\, p : X \to Y \text{ bijective}, \, o : \text{Graph(p)} \to \{1, \dots, \text{n}\} \text{ bijective}\Big\}. \tag{5.40}$$

Here the function o denotes the ordering (as in Remark 2.32) of pairs $(x, y) \in \text{Graph}(p)$, such that $p(x) = y$, where $o((x, y))$ denotes the order of the change from note x to note y. This means that a progression $p \in \mathcal{P}(X, Y)$, from Definition 4.1, now has $n!$ different variants where all voice leadings are ordered.

Example 5.31. The identity map $Id : \{1, 2\} \to \{1, 2\} \in \mathcal{P}(X, Y)$ induces two different progressions in $P_o(\{1, 2\}, \{1, 2\})$, i.e., the map o changes. We can consider following orders of the voice leading:

- The identity map $Id : \{1, 2\} \to \{1, 2\}$, with $(1, 1) \subset \text{Graph}(Id)$ first and $(2, 2) \subset \text{Graph}(Id)$ second, i.e., $o\big((1, 1)\big) = 1$ and $o\big((2, 2)\big) = 2$.
- The identity map $\{1, 2\} \to \{1, 2\}$, with $o\big((1, 1)\big) = 2$ and $o\big((2, 2)\big) = 1$.

This idea is shown in Figure 5.9, where the voices change in the given order.



(a) $o\big((1,1)\big) = 1$ and $o\big((2,2)\big) = 2$ (b) $o\big((1,1)\big) = 2$ and $o\big((2,2)\big) = 1$

Figure 5.9: Example of ordered voice leadings from Example 5.31 of the identity chord progression on $\{1,2\}$, i.e., notes $C^\sharp$ and D.

Clearly the set $\mathcal{P}_o(X,Y)$ is more complex than the set $\mathcal{P}(X,Y)$. More specifically, we can define an equivalence relation $\sim_o$ for two voice leadings $(p_1, o_1), (p_2, o_2)$ from $\mathcal{P}_o(X,Y)$ such that:

$$p_1 \sim_o p_2 \Leftrightarrow p_1 = p_2 \text{ (w.r.t Definition 2.1)} \\ \text{and } \exists \pi \in S_n, : \; o_1 = \pi \circ o_2, \tag{5.41}$$

i.e., the only difference is in the ordering of the voice leadings. Then we have that $\mathcal{P}(X,Y)$ are the equivalence classes of $\mathcal{P}_o(X,Y)$ over this relation $\sim_o$.

We have seen in Chapter 4 that $\mathcal{P}(X,Y)$ is equivalent with S_n, and now we have extra complexity by different permutations of voice leadings. This suggests that the set $\mathcal{P}_o(X,Y)$ could have some equivalence with $S_n \times S_n$. Let us consider the action of $S(Y) \times S_n$ on the set $\mathcal{P}_0(X,Y)$, for $(p,o) \in \mathcal{P}_o(X,Y)$ and $(\pi_1, \pi_2) \in S(Y) \times S_n$:

$$(\pi_1, \pi_2) \cdot (p, o) = (\pi_1 \circ p, \pi_2 \circ o), \tag{5.42}$$

such way that only π_2 changes the order. The permutation π_1 retains the original order, i.e., for $(x,y) \in X \times Y$:

$$o\Big(\big(x, \pi_1(y)\big)\Big) := o\Big(\big(x,y\big)\Big). \tag{5.43}$$

Clearly this action has as neutral element $(Id, Id) \in S(Y) \times S_n$. This action is also associative w.r.t. the group elements of $S(Y) \times S_n$.

We can prove a similar result as Theorem 4.2.

Theorem 5.32. *For any given base ordered chord progression $(p_0, o_0) \in \mathcal{P}_o(X,Y)$, we can form all other elements of $\mathcal{P}_o(X,Y)$, i.e.,*

$$\forall (p,o) \in \mathcal{P}_o(X,Y), \exists! \, (\pi_1, \pi_2) \in S(Y) \times S_n : \; (p,o) = (\pi_1, \pi_2) \cdot (p_0, o_0). \tag{5.44}$$

Proof. From Theorem 4.2, we already know that all bijections p can be formed from a unique base p_0 and permutations $\pi_1 \in S(Y)$.

It remains to show that all orderings can be made from o_0 and any permutation $\pi_2 \in S_n$. We can again make a similar argument as in Theorem 4.2. Let us consider a different ordering o_1. Since both o_0 and o_1 are bijective per definition, we can define a map $m : \{1, \dots, n\} \to \{1, \dots, n\}$ with $m\Big(o_0\big((x,y)\big)\Big) = o_1\big((x,y)\big)$. We have $o_1 = m \circ o_0$, and thus m is a bijection and is unique by uniqueness of o_0 and o_1. We thus have $m \in S_n$. □

We can show a similar result for $S(X) \times S_n$. Again, we can show that $S(Y)$ (respectively $S(X)$) is isomorphic to S_n, by (4.9). Thus, we have an equivalence with $S_n \times S_n$. But again, this is not a natural one, since the elements in X or Y are not ordered. Note the difference with ordering the elements of Graph(p), the voice leadings, and the chords itself, e.g., the elements in X.

This result is in accordance with our previous result from Theorem 5.29, as we have observed that the braids with no redundancy are shaped by the symmetric group. Obviously, not all braids that are lifted from S_n matched the description of having ordered strands, and thus the second S_n might aid in this aspect, as to define valid crossings for the braid 'shape'.

Also, the lack of redundancy in the result of Theorem 5.32, i.e., the non-redundant braids were lifts from S_n, is also explainable, since redundant crossings from Section 5.1 unnecessarily change the permutation such that π_1 no more holds its meaning, i.e., the permutation actually changes. And the second permutation, signifying the order of voice leadings, does not indicate a specific way to construct this order.

In order to consider a fully natural isomorphism, we could consider the following idea.

5.3 | Ordered Progressions of Ordered Chords

We have considered chords from the chord space $\mathbb{A}_n$, but what if we take a step back and take ordered chords $X,\ Y \in \mathbb{T}^n$? This might be useful if one wants to consider different chord inversions, where order does matter. For example, to have a specific note be the base note. The order implies then the order in frequency, i.e., the order from low to higher pitches.

Let us redefine the set of chord progressions from Definition 4.1 for $X,\ Y \in \mathbb{T}^n$. We get the following:

$$\mathcal{P}^*(X, Y) = \Big\{p : X \to Y \mid p \text{ is a bijection}\Big\}. \tag{5.45}$$

Notice that the ordering of X and Y has no impact on the actual set of elements of X and Y. Hence, $\mathcal{P}^*(X, Y)$ will just contain the same chord progression maps as in Definition 4.1, i.e., $\mathcal{P}^*(X, Y) = \mathcal{P}(X, Y)$. However, we do note that the ordering of X and Y induces a natural progression p_1. For $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, we have:

$$p_1(x_i) = y_i. \tag{5.46}$$

This natural progression p_1 implies the existence of a canonical isomorphism with S_n, since p_1 is a natural representative of the set $\mathcal{P}^*(X, Y)$. But also because the sets X and Y are ordered, there is an immediate equivalence between $S(X)$ (respectively $S(Y)$) and S_n. They are practically the same group. In Chapter 4, this was not the case since we required an arbitrary ordering of X (a class representative in $\mathbb{A}_n$) to effectively associate permutations from S_n.

In contrast with the previous part in Section 5.2, this ordering introduces no extra complexity, but rather makes the isomorphism with S_n canonical. To see that it does not distinguish between orderings of voice leadings, when considering a map $p \in \mathcal{P}(X, Y)$, one could perhaps use the ordering of X and Y to order the voice leadings. However, note that this only allows for up to two distinct orderings of voice leadings, one ordering based on starting notes in X, and one based on ending notes in Y.

Example 5.33. Take the bijections between $\{1, 2, 3\}$ and $\{1, 2, 3\}$. Clearly, these are the permutations in S_3. Consider the voice leadings of the map p with Graph(p) =

$\{(1,1),(2,3),(3,2)\}$. An ordering of voice leadings based on orderings of X and Y will result in the following two orders:

$$\begin{array}{ll} \text{based on } X: & \text{based on } Y: \\ 1 \mapsto 1, & 1 \mapsto 1, \\ 2 \mapsto 3, & 3 \mapsto 2, \\ 3 \mapsto 2, & 2 \mapsto 3. \end{array} \tag{5.47}$$

Clearly, the order on X and Y does not allow for an order like the following:

$$\begin{aligned} 2 &\mapsto 3, \\ 1 &\mapsto 1, \\ 3 &\mapsto 2, \end{aligned} \tag{5.48}$$

The latter order of changes effectively induces another permutation for the order of notes in the chords, and it thus changes the chord as an element of $\mathbb{T}^3$.

Hence, having orders on X and Y does not introduce the same complexity as in Section 5.2, where we allow to order the changes. Therefore, we have no more complexity than $\mathcal{P}(X,Y)$, but there is a natural bijection from $\mathcal{P}^*(X,Y)$ to S_n.

Similarly, If we order the voice leadings coming from ordered chords, $\mathcal{P}^*_o(X,Y)$, we get a canonical isomorphism with $S_n \times S_n$ from the following natural ordered chord progression, for ordered chords of size n:

$$\begin{aligned} 1 &\mapsto 1, \\ 2 &\mapsto 2, \\ &\dots, \\ n &\mapsto n, \end{aligned} \tag{5.49}$$

where height indicates order, e.g., the voice leading $1 \mapsto 1$ is first.

We can now try to complete the setup from Section 5.1, knowing the equivalence with $S_n \times S_n$, and try to get actual braids that represented ordered chord progressions.

5.4 | General map from $S_n \times S_n$ to B_n

As we did in Chapter 4, we can find a meaningful map from the equivalence with $S_n \times S_n$ to B_n such that the strands are ordered. We already know from Corollary 4.3 that we can get a braid with undefined crossings, i.e., the crossings do not matter, from equivalence with S_n. And from Section 5.1, we know that we can use these to make a braid with ordered strands that contain no direct redundancies. With the second permutation, we can narrow the crossing options down to a single braid.

We can use the second permutation to define an order on the strands. That is, a permutation $\pi \in S_n$ will give a total order on n strands as follows, using again the order map o:

$$o(s_{\pi(1)}) < o(s_{\pi(2)}) < \cdots < o(s_{\pi(n-1)}) < o(s_{\pi(n)}). \tag{5.50}$$

For example the identity permutation will put the first strand on top, then the second strand, then the third, ..., and so on.

This way, we have a way to construct braids from chord progressions given a base chord progression and base ordering on the chord.

Example 5.34. We consider ordered chords and ordered voice leadings on the chords such that we have a clear natural base progression. We can now consider the chord progressions from G^7 to C^{maj7}, i.e., (G, B, D, F) to (C, E, G, B) in this order. The most efficient, non-crossing voice leading is that of:

$$G \mapsto G, \qquad B \mapsto B, \qquad D \mapsto C, \qquad F \mapsto E. \tag{5.51}$$

Since we work with ordered voice leadings, let us consider the order as given in (5.51). We have as a base progression, from the order of the individual notes:

$$\begin{aligned} (1) \;\; G \mapsto C \;\; (1), \\ (2) \;\; B \mapsto E \;\; (2), \\ (3) \;\; D \mapsto G \;\; (3), \\ (4) \;\; F \mapsto B \;\; (4), \end{aligned} \tag{5.52}$$

such that the first permutation, marking the used voice leading, becomes:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \tag{5.53}$$

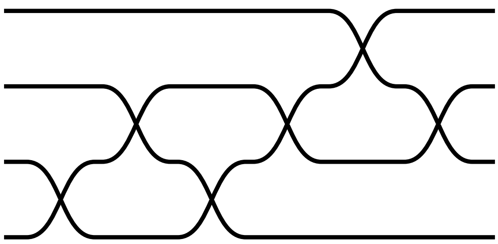
in the two-line permutation notation. For example, this means that in (5.52), the note C (in C^{maj7}, with the right index) permutes with G such that we have $G \mapsto C \mapsto G$ as in the wanted voice leading in (5.51). This is the permutation (13)(24) in cycle notation. By Theorem 2.12, we can write the permutation in (5.53) as the permutation (12)(23)(12)(23)(34)(23). Written this way, it will be easier to construct the braid.

For the order of the voice leadings we now use the permutation to switch the voice leadings around, using the left index in (5.52). We get the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \tag{5.54}$$

which is the identity permutation because we just used the given order of notes in G^7 to form our ordered voice leading in (5.51).

By Corollary 4.3, we now transform the permutation (12)(23)(12)(23)(34)(23) into the following braid:

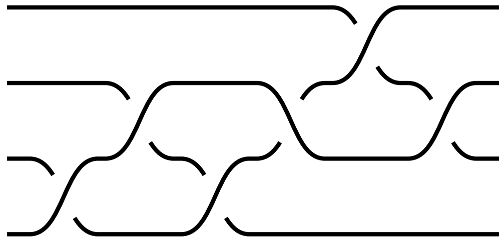


(5.55)

with undefined crossings.

Now we can use the identity permutation from (5.54) to define the crossings. For example, since strand 1 is on top ($Id(1) = 1$ in the identity permutation), we get that the first crossing is an overcrossing.

We conclude with the following braid in B_4, with four strands signifying every voice leading.

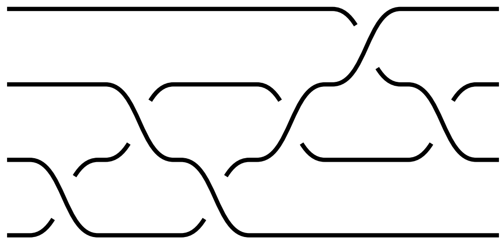


(5.56)

If we had, instead of the identity permutation as order, the following arbitrary permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \tag{5.57}$$

we get the braid from (5.55) with crossings as in the following braid.



(5.58)

The final braids in (5.56) and (5.58) are both knot-isotopic to the Hopf link, and when applying Seifert's algorithm, we end up with a Seifert surface with genus one.

Remark 5.35. Notice that from the first given order permutations in Example 5.34, we cannot distinguish the full order from the braid itself. In the braid in (5.56), we have that strands three and four do not cross each other, making it impossible to know their relation (as it is also not implied by transitivity). This braid could be the result of the following ordered voice leadings:

$$G \mapsto G, \qquad B \mapsto B, \qquad D \mapsto C, \qquad F \mapsto E, \tag{5.59}$$

$$G \mapsto G, \qquad B \mapsto B, \qquad F \mapsto E, \qquad D \mapsto C, \tag{5.60}$$

where the last two voices (three and four) are swapped, as we do not know what order they follow.

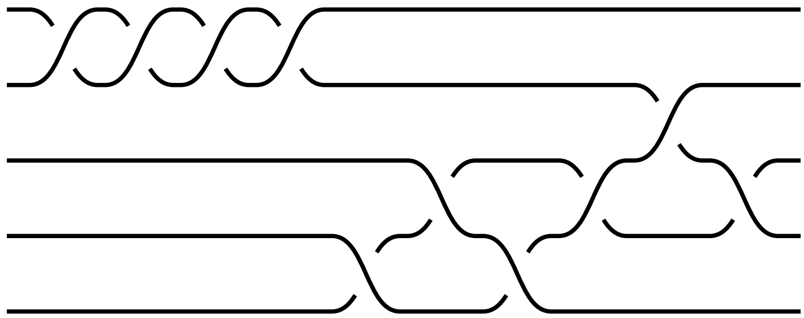
This is however not the case with the second permutation resulting in the braid in (5.58) where the entire order of strands is visible in the braid.

Clearly, the braid does not always embody the full total order, and thus the map $S_n \times S_n \to B_n$ that we outlined in this section will not be injective because it can lose information on the total order.

Just as in Chapter 4, we have just the braid shapes from B_n/P_n, without the pure braids, as it is equivalent with S_n. But in the context of ordered voice leadings, this also makes sense as we have already shown in Corollary 5.13 that the only pure braid that abides to an order on the strands is the trivial braid 1. And thus to imagine a braid like $\sigma_1^2\sigma_2$ is not viable since it contains a non-trivial pure braid σ_1^2. In the quotient group B_n/P_n the braid $\sigma_1^2\sigma_2$ is equivalent to σ_2, which is a viable braid. But this also means we cannot imbed the chord distance in the braid as we did in Chapter 4 in (4.11) as pure braids of the form σ_1^{2k}

are not the trivial braid 1, except for a distance of $k = 0$. That is, if we wish to have this part of the braid to be in accordance with a valid braid from Section 5.1. We can choose to define our braids in such a way that only the second part of the braiding, coming from $S_n \times S_n$ is ordered in the same fashion as Section 5.1, and the first part signifying the distance is irrelevant to that.

Example 5.36. With the chord progression from Example 5.34, we had the braid coming from $S_n \times S_n$ depicted in (5.58). Adding a braid showing the chord distance we get the following braid:



(5.61)

From (5.51), we clearly have a difference of 2 (with the L^1 norm equipped) between the chords G^7 and C^{maj7}, and this is easily visible by the four crossings of the fourth strand with the fifth strand.

6 | Digression: Partial and Singular Chord Progressions

Up until now, we have only considered chord progressions $p : X \to Y$ between two chords X and Y of equal size, but in reality chords do not have to be the same size. We introduce some ways to consider progressions between chords of different sizes and more general ways to define voice leadings with partial singular braids. This chapter is meant more to start a discussion and spark interests. Due to timing, we will leave this as a fun addition to this thesis, touching on some more general topics and other ways that can be pursued. We start with looking into the definition of partial singular braids

6.1 | Partial Singular Braids

One can define a more general notion of braids, such that collisions are allowed. A collision in a braid is a singular point. Other types of braids can contain only k strands with $k < n$, while accounting for n strands. With this in mind we can talk about sub braids, i.e., a subset of the strands and their braiding. The combination of these two notions, singularity and partial braids, gives rise to a monoid (see Definition 2.13). But first, we must define a partial permutation:

Definition 6.1. *The* ***symmetric inverse monoid*** *is a monoid that contains all partial permutations of size* n*. This is the set of all permutations that permute* k *out of* n *elements, with* $k \leq n$*. The monoid is denoted* $\mathcal{I}_n$*. Elements can be visualised by tableau notation, with a diamond '*$\diamond$*' signifying no permutation.*

Example 6.2. An example of a partial permutation of size $n = 5$ is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & \diamond & 1 & \diamond \end{pmatrix} \tag{6.1}$$

where the value of 1 permutes to 4, 2 to itself, 4 to 1, and the rest does not get permuted. Intuitively, this can be interpreted as forming a three-letter word from a set of five letters.

Now we can define singular and partial braids as follows:

Definition 6.3. *The set of* ***singular braids*** $\mathcal{SB}_n$ *is the monoid with the regular braid generators* σ_i *and their relations and an extra generator* τ_i *signifying a singular crossing with the following additional relations:*

$$\sigma_i\tau_j = \tau_j\sigma_i, \quad |i-j| > 1, \tag{6.2a}$$
$$\tau_i\tau_j = \tau_j\tau_i, \quad |i-j| > 1, \tag{6.2b}$$
$$\sigma_i^{\pm 1}\tau_i\sigma_i^{\mp 1} = \tau_i \qquad \forall i < n \tag{6.2c}$$
$$\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}, \quad \forall i < n, \tag{6.2d}$$
$$\sigma_i\sigma_{i+1}\tau_i = \tau_{i+1}\sigma_i\sigma_{i+1}, \quad \forall i < n, \tag{6.2e}$$
$$\sigma_i\tau_{i+1}\sigma_i = \sigma_{i+1}\tau_i\sigma_{i+1}, \quad \forall i < n. \tag{6.2f}$$

Definition 6.4. *The set of* ***partial braids*** $\mathcal{IB}_n$ *is the* ***inverse braid monoid*** *with the regular braid generators* σ_i *and their relations and an extra generator* ε_i *with* $i \in \{1, \dots, n\}$

signifying an identity element for all strands but the ith strand, i.e., it removes the ith strand. We have the following additional relations:

$$\sigma_j\varepsilon_i = \varepsilon_i\sigma_j, \quad |i-j|>1, \tag{6.3a}$$
$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i, \quad \forall 1 \le i,j \le n, \tag{6.3b}$$
$$\varepsilon_i\sigma_i = \sigma_i\varepsilon_{i+1}, \quad \forall 1 \le i < n, \tag{6.3c}$$
$$\sigma_i\varepsilon_i = \varepsilon_{i+1}\sigma_i \quad \forall 1 \le i < n. \tag{6.3d}$$
$$\varepsilon_{i+1}\sigma_i^2 = \sigma_i^2\varepsilon_{i+1} = \varepsilon_{i+1}^2 = \varepsilon_{i+1}, \quad \forall 1 \le i \le n, \tag{6.3e}$$
$$\sigma_i\varepsilon_i\varepsilon_{i+1} = \varepsilon_i\varepsilon_{i+1}\sigma_i = \varepsilon_i\varepsilon_{i+1}, \quad \forall 1 \le i < n. \tag{6.3f}$$

The set of partial singular braids is the following:

Definition 6.5. *The monoid $\mathcal{PSB}_n$ is the set of equivalence classes of partial singular braids. Here, braids are equivalent if they induce the same partial permutation and are sub-braids of equivalent singular braids[10].*

For braid $\beta \in \mathcal{PSB}_n$, we assume following notations:

- *$|\beta|$ is the number of used strands.*
- *$N(\beta)$ is the number of singular points in the braid.*

The operator of this monoid is a concatenation where strands are kept if they connect from one braid to another, i.e. for braids β_1, β_2, a strand i_1 in β_1 is kept if there is a strand j_2 in β_2 that starts at the vertex where i_1 ends, and vice versa.

Thus, concatenating two braids without the strands that end up disappearing due to the above process will result in the same braid as the concatenation with those extra strands.

As mentioned in the definition, partial braids induce a partial permutation. This is done by an epimorphism, i.e., a surjective homomorphism [9]. Note the similarities between this and the homomorphism M from Definition 2.42.

The idea of singular crossings also induces the idea of equivalent singular knots by the respective singular Markov moves. Note that we have $\mathcal{SB}_n = \{\beta \in \mathcal{PSB}_n \mid |\beta| = n\}$.

Example 6.6. An example of partial singular braids is shown in Figure 6.1, where two braids are depicted with one being a sub-braid of the other. The smaller and larger braid induces partial permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & \diamond & \diamond & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 3 & 5 \end{pmatrix} \tag{6.4}$$

respectively.

For more information on this topic, one may refer to [10, 9, 11, 24].

6.2 | Regular Braids for Chord Progressions of Different Sizes

In a musical context, when we consider a voice leading from a chord $X \in \mathbb{A}_n$ to $Y \in \mathbb{A}_m$, with $m < n$, there will be voices that lead to the same note.

Example 6.7. Consider a voice leading from a G^7 chord to a C chord. These chords have, respectively, the notes (G, B, D, F) and (C, E, G). A nice voice leading could be $G \mapsto G$, $B \mapsto C$, $D \mapsto E$, $F \mapsto E$. Here, the note E appears twice. We project the first chord onto the size of the second chord.

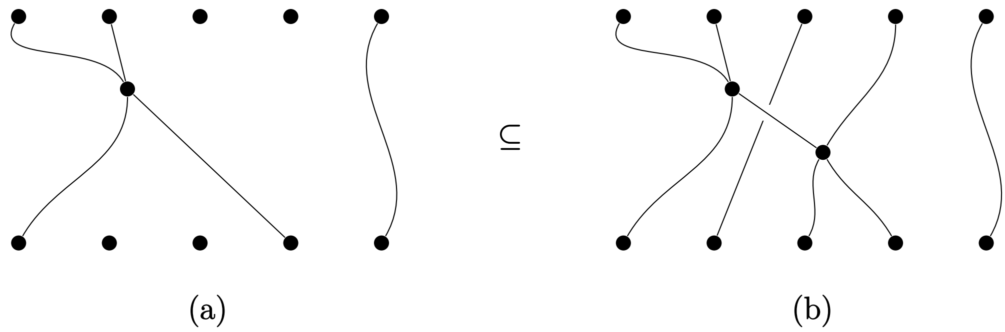


Figure 6.1: Two partial singular braids in $\mathcal{PSB}_5$, with braid (a) being a sub-braid of the singular braid in (b).

Mathematically, this boils down to a surjective, non-injective map $p : X \to Y$.

Equivalently, we can consider chord progressions from a chord $X \in \mathbb{A}_n$ to $Y \in \mathbb{A}_m$, with $m > n$. Here, there will be a note from the first chord X that appears twice.

Example 6.8. Consider a voice leading from a G chord to a C^{maj7} chord. These chords have, respectively, the notes (G, B, D) and (C, E, G, B). A nice voice leading could be $G \mapsto G$, $B \mapsto B$, $D \mapsto E$, $B \mapsto C$. Here, the note B appears twice. We extend the first chord to match the size of the second chord.

It is not possible to express this extension as a function since we have two images for one point in X. We could consider the 'inverse' voice leading from Y to X and again get surjective, non-injective maps.

These are two examples where we extend the smaller chord up to the bigger chord, but one can also project the bigger chord up to the smaller chord.

Example 6.9. In the above examples, one could perhaps drop the fifth in the larger chords to get the following voice leadings, respectively:

$$(G, B, F) \to (C, E, G), \tag{6.5}$$

and

$$(G, B, D) \to (C, E, B). \tag{6.6}$$

With just the one note difference, this does not have too much of an effect on the chords, but when considering larger differences, information on the chords surely can be lost by dropping notes.

We consider extensions and projections because the regular Artin braids do need an equal amount of strands. Obviously, we can consider specific chord progressions between the chord's projections and extensions by Chapter 4 and Chapter 5, but this does not imply anything for the other possible projections and extensions. Hence, we cannot just copy and paste previously acquired knowledge on chord progressions without losing information on the other possible projections and extensions of chords. We have to consider all projections and extensions. Let us define the following sets as in [16]:

Definition 6.10. *The **projection set** of a chord $X \in \mathbb{A}_n$ is the following set:*

$$\mathrm{Proj}_m(X) = \left\{X' \subset X \,\middle|\, |X'| = m\right\}, \tag{6.7}$$

with $|X'|$ the size of the sub-chord X', i.e., we have $m < n$.

Analogously, we define:

Definition 6.11. *The **extension set** of a chord $X \in \mathbb{A}_n$ is the following set:*

$$\text{Ext}_m(X) = \left\{ X' \supset X \,\middle|\, |X'| = m \text{ and } X' \text{ contains only elements of } X \right\}, \tag{6.8}$$

with $|X'|$ the size of the super-chord X', i.e., we have $m > n$.

With that in mind, we could define the following set

Definition 6.12. *Define the set of **projection chord progressions** as follows:*

$$\mathcal{P}_{n,m}(X,Y) = \bigcup_{X' \in \text{Proj}_m(X)} \mathcal{P}(X',Y), \tag{6.9}$$

with $n > m$.

Equivalently, we define:

Definition 6.13. *Define the set of **extension chord progressions** as follows:*

$$\mathcal{P}_{n,m}(X,Y) = \bigcup_{X' \in \text{Ext}_m(X)} \mathcal{P}(X',Y), \tag{6.10}$$

with $n < m$.

These definitions do give a way to get back to Chapter 4 and Chapter 5. Since we distinguished between different chords in a progression by adding the distance between chords in the braid, we can distinguish between different extensions and projections.

Example 6.14. When considering again the chords from Example 6.7, we can consider the following two projection voice leadings that have the same chord distance to the second C chord.

$$\begin{array}{cc} (1) & (2) \\ G \mapsto G, & G \mapsto G, \\ D \mapsto C, & B \mapsto C, \\ F \mapsto E, & D \mapsto E. \end{array}$$

The order of occurrence is not important here.

These two projections will give a different permutation though.

If two different projections or extensions have the same distance and permutation to the second chord, then it makes sense that they result in the same braid, as they do behave in very similar ways.

6.3 | Partial Singular Braids

Up until now, we have only considered regular braids, i.e., braids in the group B_n. But one can also introduce partial (singular) braids as defined in Definition 6.5. In Corollary 4.3, we have concluded that a single crossing type collapses the braid structure to that of the symmetric group S_n. However, the final representation in [6] suggests using singular crossings to identify crossings of voice leadings. This also makes some sense since a singular crossing denotes that strands collide and are at some point truly equal, which nicely fits the idea of voice leadings that hit the same note. Voices that cross are always on the same note, and thus always collide in this manner. Thus, for voice crossings we can choose all crossings to be singular crossings. A musical piece with n voices will be in the following sub-monoid of $\mathcal{SB}_n$.

$$\mathcal{T}_n := \langle \tau_i \mid 1 \leq i < n,\ \tau_i\tau_j = \tau_j\tau_i \text{ for } |j-i| > 1\rangle. \tag{6.11}$$

Recall the notation $\langle\cdot\rangle$ from Definition 2.8. Clearly this sub monoid has less structure than the braid monoid containing only braids with positive crossings B_n^+ (respectively B_n^-) as it lacks a relation of the form $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$, which is not implied by the relations in (6.2). This implies that $\mathcal{T}_n$ will contain more *braid shapes*[20] than in B_n. If we were to map these singular braids into B_n by a map like the following:

$$\mathcal{T}_n \to B_n^+ : \tau_i \mapsto \sigma_i, \tag{6.12}$$

or equivalently $\mathcal{T}_n \to B_n^- : \tau_i \mapsto \sigma_i^{-1}$, then we essentially lose information from the monoid $\mathcal{T}_n$ as we introduce the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. This also shows why the naive representation in Section 3.1 is not a good one, as the above map is clearly surjective but not injective.

And if we really want to go deeper into this, one could introduce another generator. As is visible in the example in the Appendix A, the voices can collide without crossing, and collide with crossing, i.e., one induces an identity permutation and the permutation $(i\ i+1)$.

We introduce the following monoid:

$$\mathcal{T}_n^* = \langle \tau_i^1, \tau_i^2 \mid 1 \leq i < n,\ \tau_i^{1,2}\tau_j^{1,2} = \tau_j^{1,2}\tau_i^{1,2} \text{ for } |j-i| > 1\rangle, \tag{6.13}$$

such that the expansion of the map M from Definition 2.42 becomes $M(\tau_i^1) = Id \in S_n$ and $M(\tau_i^2) = (i\ i+1) \in S_n$. Clearly, we have that $\mathcal{T}_n$ is a quotient monoid of $\mathcal{T}_n^*$ over the relation $\tau_i^1 \sim \tau_i^2$.

In [6], to deal with chord progressions between chords of different sizes, the author used partial permutations. However, the partial permutations introduced in [6] are not exactly elements of $\mathcal{I}_n$ since they allow two different elements to permute to the same non-empty element[21]. The author defines partial maps on multisets of the chords. An example of what such a permutation looks like is given below.

Example 6.15. Considering again the chord progression from Example 6.7, G^7 to C, we can get the following partial permutation:

$$\begin{pmatrix} G & B & D & F & C & E \\ G & C & E & E & \diamond & \diamond \end{pmatrix}. \tag{6.14}$$

[20] We mean braids independent of the crossing type, like braids in $B_n/\sim$ as in B_n/P_n but including pure braids. Literally the crossings that occur without regarding the crossing type.

[21] Not the $\diamond$ element.

Notice that we have taken the order of the notes as they were provided in Example 6.7. Two notes, D and F, permute to E which is not possible unless using multisets.

We can define the partial permutations in a way that does not use multisets but makes for distinguishable elements in the set of partial braids. Contrary to the approach in Section 6.2, where all different extensions or projections basically give the same set of progressions. Recall that we define those sets of progressions as in Definition 6.12 and Definition 6.13, where we have the union of sets $\mathcal{P}(X', Y)$ as in Definition 4.1. By using partial braids, we can distinguish between those different projections by just using the partial permutation between the projection and the end chord. We have the following example.

Example 6.16. For the chord progression G^7 to C, i.e., (G, B, D, F) to (C, E, G), we could get the partial permutation:

$$\begin{pmatrix} G & B & D & F & C & E \\ G & C & \diamond & E & \diamond & \diamond \end{pmatrix}, \tag{6.15}$$

or another projection could give us the following permutation:

$$\begin{pmatrix} G & B & D & F & C & E \\ G & C & E & \diamond & \diamond & \diamond \end{pmatrix}. \tag{6.16}$$

Notice that we again need to include all relevant notes in both chords to get an actual permutation of said notes.

As also suggested in [6], the order of notes is again important here. Considering extensions, as in Definition 6.13, does not fit well in this idea since we cannot sample k elements out of n with $k > n$. This will result in choosing to drop $m - n$ notes from the destination chord to be a meaninful partial permutation. But this defeats the purpose of considering extended chord progressions.

Since partial braids induce a partial permutation (recall Definition 6.5), we can again try to lift the this partial permutation to a partial braid, i.e., we can try to find a map

$$\mathcal{I}_n \to \mathcal{PSB}_n. \tag{6.17}$$

The issue with crossing types will be the same as in Chapter 4 and Chapter 5. However, do note that we can also allow again for singular crossings, and as already suggested before, these might aid in the dispute on the crossing types. We could thus define these braids as partial singular braids with only singular crossings.

Example 6.17. Consider again the chord progression G^7 to C, with the permutation:

$$\begin{pmatrix} G & B & D & F & C & E \\ G & C & \diamond & E & \diamond & \diamond \end{pmatrix}. \tag{6.18}$$

We get the first partial braid displayed in Figure 6.2. Notice that there are no crossings made, and thus there are no singular points. Considering the same chord progression but a different projection, with permutation:

$$\begin{pmatrix} G & B & D & F & C & E \\ G & C & E & \diamond & \diamond & \diamond \end{pmatrix}, \tag{6.19}$$

yields the second partial braid displayed in Figure 6.2.

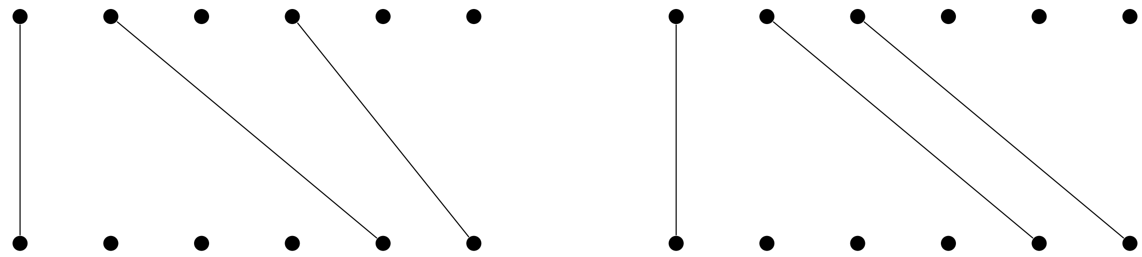


Figure 6.2: The partial braids inducing the partial permutations from Example 6.17

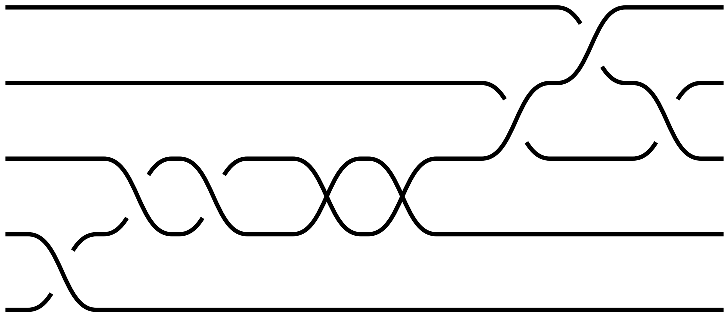


Figure 6.3: Singular braid that abides by a non-strict order of the strands. With singular operator τ_i, the braid word is $\sigma_1^{-1}\sigma_2^{-2}\tau_2^2\sigma_3\sigma_4\sigma_3^{-1}$.

In Section 5.1, we defined a strict order on the strands. We could make this order non-strict by allowing singular crossings representing strands that have the same order-value.

We mentioned before in this section that antisymmetry was not possible since we considered all strands strictly different. But now we could have such a case where we allow two strands that are on the same level to cross over and under each other in a way that was not allowed before. This thus means that all of our restrictions in Section 5.1 do not apply to strands that cross each other singularly.

Also, we can say that restrictions to the linking number now also apply universally to collections of strands with singular crossings, i.e., when crossing one strand of a singular duo positively, we also need to cross the other in the singular duo positively or not (and similarly for negative crossings). If strands s_1 and s_2 collide singularly, then for strand s_3 we get the following condition:

$$\operatorname{sgn}(l_{B_n}(s_1, s_3)) = \operatorname{sgn}(l_{B_n}(s_2, s_3)), \tag{6.20}$$

with sgn the sign number. Those linking numbers were already bounded by Lemma 5.10, and now they have to have the same sign. An example is shown in Figure 6.3. Here strands 1 and 3 have singular crossings and are thus equal in order. Clearly, the crossings σ_2^{-2} normally induce a contradiction, but since we allow singularities, these strands are the same in order by antisymmetry. This is also clear by the following singular crossings. The order of the strands is the following one:

$$o(2) < o(1) = o(3) < o(5) < o(4), \tag{6.21}$$

where o is again the order.

In theory, we have no need for the singular crossings if something like σ_2^{-2} occurs. By antisymmetry, we know that the strands will have the same order, and thus we do not need the extra singular crossing to confirm this. This results in the braid shown in Figure 6.4.

Allowing for more complex linking thus allows for more interesting braids, contrary to having a strict order as in Section 5.1. Clearly, allowing more strands to be on a single

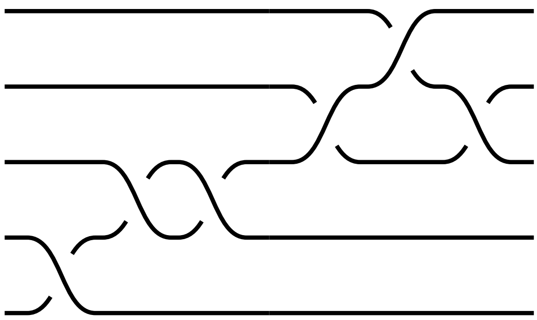


Figure 6.4: Singular braid that abides by a non-strict order of the strands. The braid word is $\sigma_1^{-1}\sigma_2^{-2}\sigma_3\sigma_4\sigma_3^{-1}$.

level, allows for more crossings to occur that previously were not allowed. If all strands are to be considered on a similar level, we just get the set of B_n as valid braids, since there are no restrictions on crossings within one order level.

7 | Conclusions

With the literature from dr. Bergomi, we pondered upon the potential of representing voice leadings and chord progressions as braids. In Chapter 3, we considered two different representations for voice leadings and chord progressions respectively. We identified three issues with these representations, and another as a consequence:

Issue 1 For both representations in Section 3.1 and Section 3.2, we had a lack of distinction between over- and undercrossings. Something that is valuable in the braiding group. As a result of Corollary 4.3, we have that B_n, under the equivalence of identifying over- with undercrossings, is equivalent to the symmetric group S_n.

Issue 2 The representation in Section 3.2 was dependent on choosing a base note in order to construct a braid. Since the representation considered chords to have pitches in the space $\mathbb{T}$ from Definition 3.4, the representation suffers from having no origin. This happens because $\mathbb{T}$ is a torus.

Issue 3 Another issue with the model of Section 3.2 was that chord progressions have all changes occur simultaneous, but this behaviour was not reflected in the braids where the order of the changes did seem to matter.

Issue 4 There is no natural lift that maps elements of S_n to B_n. If we wish to have a natural map from our musical elements to B_n, we need to have some consistency in the possible images.

Below, we shall briefly discuss these issues, how we approached them, and reflect to which extent our approaches were successful.

Issue 1

This issue is observed by Theorem 4.2 and Corollary 4.3 in Chapter 4. The result of Corollary 4.3 states that a braiding representation without distinction between over- and undercrossings results in an equivalence with S_n, which has significantly less structure than B_n.

The Theorem 4.2 sheds some light on why this happens for voice leadings. Recall that the set of all voice leadings $\mathcal{P}(X, Y)$ from Definition 4.1 is a torsor of the symmetric group (recall Definition 2.18). Recall the definition of the set $\mathcal{P}(X, Y)$:

$$\mathcal{P}(X, Y) = \left\{ p : X \to Y \mid p \text{ is a bijection} \right\}. \tag{7.1}$$

Since we took chords X and Y in $\mathbb{A}_n = \mathbb{T}^n / S_n$, where order does not matter, we get that there is no initial reference point for these chords X and Y. As a consequence, we found more than one equivalence with the symmetric group $S(Y)$ or equivalently $S(X)$. The equivalence with the symmetric group again is clear by considering the composition of elements p of $\mathcal{P}(X, Y)$ with permutations π of $S(Y)$ or of $S(X)$. This composition results in other elements of $\mathcal{P}(X, Y)$, as shown in Theorem 4.2. The lack of origin we talked about is embodied here by the chosen element $p \in \mathcal{P}(X, Y)$.

With an equivalence to the permutations of $S(X)$ or $S(Y)$, one could also draw an equivalence to S_n, with n the size of both X and Y. With X and Y being unordered, we get that there are many isomorphisms with S_n. An element in S_n essentially orders the elements. By considering equivalences with S_n, we essentially again create a distinction in the order of elements.

We had that an element $p \in \mathcal{P}(X,Y)$ is equivalent with an element $\pi \in S(Y)$. This was based on some base element $p_0 \in \mathcal{P}(X,Y)$. And if we have a map $b : Y \to \{1,\dots,n\}$ that orders Y, we get an equivalence with S_n.

Thus, ordering either X or Y, i.e., taking it as element of $\mathbb{T}^n$ (from Definition 3.4) rather than $\mathbb{A}_n$, gives a canonical isomorphism from $S(X)$ or $S(Y)$ to S_n. We would still require a non-canonical base chord progression p_0. Thus, by ordering both X and Y, taking them as elements of $\mathbb{T}^n$, we solve this issue as well by introducing the natural chord progression that maps each element in X to its respective order element in Y, i.e, the map p_0 becomes:

$$p_0 : X = (x_1,\dots,x_n) \to Y = (y_1,\dots,y_n) : x_i \mapsto y_i. \tag{7.2}$$

Hence, ordering both X and Y gives an immediate canonical equivalence from an element in $\mathcal{P}(X,Y)$ to a permutation in S_n. However, this does not work the other way around. Having a natural progression p_0 does not imply orders on X and Y.

Also, notice that assuming an arbitrary natural base progression $p_0 \in \mathcal{P}(X,Y)$ and ordering either X or Y is equivalent with ordering both X and Y, since orders can be transferred by the natural bijection p_0 between X and Y.

Considering Theorem 4.2 and Corollary 4.3, we have that the voice crossing representation of Section 3.1 is equivalent with S_n. This makes sense since voice crossings are essentially permutations of pitches. Also, with regard to Theorem 4.2, we need to consider the actual picthes as elements of $\mathbb{R}$ rather than $\mathbb{T}$. This clearly considers X and Y as ordered for the voice leading with voice crossings. Hence, there is an immediate isomorphism with S_n.

Issue 2

The occurrence of Issue 2 is something that we did not resolve. Our methods also suffer from this issue. Our 'circles with no origin' are the elements X and Y being in $\mathbb{A}_n$ rather than $\mathbb{T}^n$.

The model in Section 3.2 does limit the number of different representations to 12, considering all 12 notes from (2.1) as roots, while the number of our different representations is dependent on the size of chosen chords X and Y.

Issue 3

The happening of Issue 3 induces the idea that order of voice leading note changes does matter, mathematically. Musically, ordering changes is like an arpeggio where changes occur in sequence rather than all at once.

In Section 5.2, we proved that voice leadings with ordered changes, e.g., arpeggios, are a torsor of the group $S(Y) \times S_n$, i.e., an equivalence without origin, dependent on a chosen base progression p_0. While the order aspect involves an ordering, there is again no natural equivalence with S_n. This is clear if we consider an element $p \in \mathcal{P}(X,Y)$ and a permutation $\pi \in S_n$. Since there is no order on either X, Y, or the voice leadings themselves (the elements of $\mathrm{Graph}(p)$), we cannot identify the permutation π with an order of said elements of p. Here, we require both a base progression p_0 and a base order o_0.

Yet, as we mentioned with Issue 1, having a natural progression p_0 with a defined order on the changes will be equivalent with having an order on both X and Y. Let us consider an element $(p,o) \in \mathcal{P}_o(X,Y)$ ((5.40)). This will induce an order on both X and Y as follows:

$$O^X_{(p,o)} : X \to \{1,\dots,n\} : x \mapsto i \qquad \text{such that } o((x,p(x)) = i, \tag{7.3}$$

and similarly for $O^Y_{(p,o)} : Y \to \{1, \ldots, n\}$ with $o(p^{-1}(y), y) = i$. This results in a natural equivalence with S_n.

However, notice that the set $\mathcal{P}_o(X,Y)$, with ordered changes, was equivalent with $S(Y) \times S_n$ and not just S_n, by Theorem 5.32. This is because in the set $\mathcal{P}_o(X,Y)$ we allow all different orders, and the above mentioned equivalence is based on a fixed order. More specifically, we can make the following map:

$$\mathcal{P}_o(X,Y) \to \Big\{f : \mathcal{P}(X,Y) \to S_n \,|\, f \text{ is bijective}\Big\} : \tag{7.4}$$

$$(p_0, o) \mapsto \Big(f : p \mapsto O^Y_{(p_0,o)} \circ p \circ (O^X_{(p_0,o)})^{-1}\Big), \tag{7.5}$$

with $O^X_{(p_0,o)}$ and $O^Y_{(p_0,o)}$ as in (7.3). Thus elements of $\mathcal{P}_o(X,Y)$ induce natural equivalences between $\mathcal{P}(X,Y)$ and S_n.

Issue 4

The occurrence of Issue 4 was also described in Section 4.1. The idea of ordering changes solves the issue of having no crossings defined. In Section 5.4, we have constructed a map that embeds the ordered voice leading into B_n. However, this map is not injective. In Remark 5.35, we had a clear example of this.

Throughout Section 5.1, we have tried to get a grasp on the image of the map in Section 5.4. As this map suggests, we consider an order on the voices as an order of the strands. This order on the strands can define our crossings in a meaningful way such that the actual order is visible from the braid, though not entirely as we mentioned in Remark 5.5. This order is visible by layering the strands, i.e., if a voice has a higher order than another, its respective strand will also always stay above the other respective strand. However, this puts some limitations on the crossings that can occur, recalling Example 5.1. Throughout Section 5.1, we have tried to identify these limitations, based on subsets of the braiding group B_n such as the positive or negative permutation braids, and by bounds on knot invariants.

We first found a bound on our own defined *braid linking number*, Definition 5.7, which we later used to put a bound on the linking number knot invariant in Theorem 5.11. Also, we have bounded the knot genus invariant by using the formula for surface genus in (2.36) from [23]. Both these bounds are also dependent on the amount of voices that are considered, and thus the amount of strands.

The bounds on the linking number and the knot genus tell us the exact limitation we have for braids with a valid order on the strands, and thus on the voices. While it is hard to identify all the knots that abide to these limits, we clearly have that for braid groups there is a significant amount of braids that do not abide to these limits. However, as in Figure 5.1, some knots can be embodied by valid braids when considering more strands.

Having a valid braid representation as a knot is not a knot invariant, as we have seen in Figure 5.1. However, we have proven with Lemma 5.3 that having a valid order on the strands is a braid invariant for equivalent braids in B_n.

Then, we have seen that it is quite difficult to consider all valid braids since we can have redundant crossings (Definition 5.27) that give redundant information on the voice order but change the braid. This makes for artificial complexity by redundancy, which is unwanted since a voice leading with a particular order could then be represented by many different braids. Hence, we considered just those braids without those redundancies. In Theorem 5.29, we showed that these non-redundant braids can be formed by lifting elements of S_n to B_n. Obviously, there is no unique lift from S_n to B_n, as we have seen in Example 5.34 where two different braids with ordered strands are provided that are equivalent in B_n/P_n, another isomorphism of S_n. Since they do give different braids in B_n, we can conclude that there is no unique lift that maps to all the valid braids.

With the definition of redundancy, having no redundancy limits the number of crossings between two strands to just one, just like the positive permutation braids SB_n (Definition 4.5).

We were also able to prove that the most efficient type of braids, those that order the strands totally in a minimal required number of crossings, are knot-isotopic with the unknot. While being a rather small subset of the valid braids, this implies that the set of valid braids might not be that interesting.

In this thesis, we have tried to find a natural way to represent voice leadings and chord progressions as braids. Not every map from voice leadings to braids will be natural in the sense that it is meaningful in this musical context. This is exactly the difficulty that we struggle with. If we wish to have a map that makes sense musically, we need to choose carefully what exactly this map to braids will embody, since what it embodies will have to be meaningful.

We have concluded with some natural maps to braids by only considering what is musically meaningful for braids, which is mostly just permutations. Hence, our model does not distinguish very well between different pitches, i.e., one could easily take a voice leading and change one note a semitone up and still have the same result. This is because the model is primarily based on the permutation, which is independent of the absolute pitches, and more related to their relative ratio. We have, however, tried to resolve this by trying to embed the natural metric of the chord space into an extra knot component that links with the initial braid, depending on the distance. This works well, but will be in contradiction with ordering the strands, if one would want to consider this over the entire braid.

7.1 | Recommendations

For those interested in continuing some of the ideas discussed in this thesis, we suggest the following. Since we consider notes to be elements of the pitch space $\mathbb{T}$, it might be interesting to actually consider braids on this space, i.e., the braiding group $\pi_1(\mathrm{Uconf}_n(\mathbb{T}))$. This means that strands start on a circle and evolve over a cylinder. Since it is on a circle, the strands can easily permute in position without actually crossing, by going around the circle. However, this space lacks a second dimension to specifically allow over- and undercrossings. This might make it not as interesting. However, one could always consider just singular crossings, as in Chapter 6.

Furthermore, some suggestions from Chapter 6 could be more interesting to pursue. The suggestions from Chapter 6 allow for more general braids, and can thus perhaps allow for more musical meaning to be transferred to braids. Recall, for example, the singular crossings (Definition 6.3) which make more sense musically. Partial singular braids capturing more musical meaning make for a better representation of the music.

Also, we have only discussed bounds for two knot invariants but there are many more. There are knot invariants defined in the general linear group, i.e., as invertible matrices, or as polynomials. Some of these knot invariants are much more used than the linking number or knot genus, and thus are perhaps more interesting to look into.

Also when considering singular braids, one could look into Vassiliev knot invariants.

These are invariants that are extended from regular invariants. When considering singular crossings, the Vassiliev invariants assume both under- and overcrossings in its stead, and take the difference of the resulting initial knot invariants. This is used recursively for all singular crossings.

When considering the braids with a valid order, it might be useful to investigate wether the amount of admissible braids is finite or not. The braid group itself is infinite, but as we saw the valid braids are a smaller subset of all braids. When we have an estimate of how many such braids are possible, we could have a better grasp as well on what kind of braids these are. This is mostly regarding the valid braids with redundant crossings, as it is difficult to say how many redundancies one can introduce to effectively alter the braid every time.

If you wish to just consider the musical aspects of this thesis, you could look into metrics for the symmetric group [17]. Since our abstract results solely included variations of permutations, having a metric allows to analyse the voice leadings in this way.

ARTIFICIAL INTELLIGENCE STATEMENT

In this thesis, artificial intelligence (AI) was used to evaluate written text, as well as generating or aiding in writing LaTeX-code for some of the more complex Tikz figures such as Figure 2.1, Figure 6.1, Figure A.1, MusiXTEX figures such as Figure 2.2 and Figure 5.9, and the code for generating Figure 3.3.

8 | References

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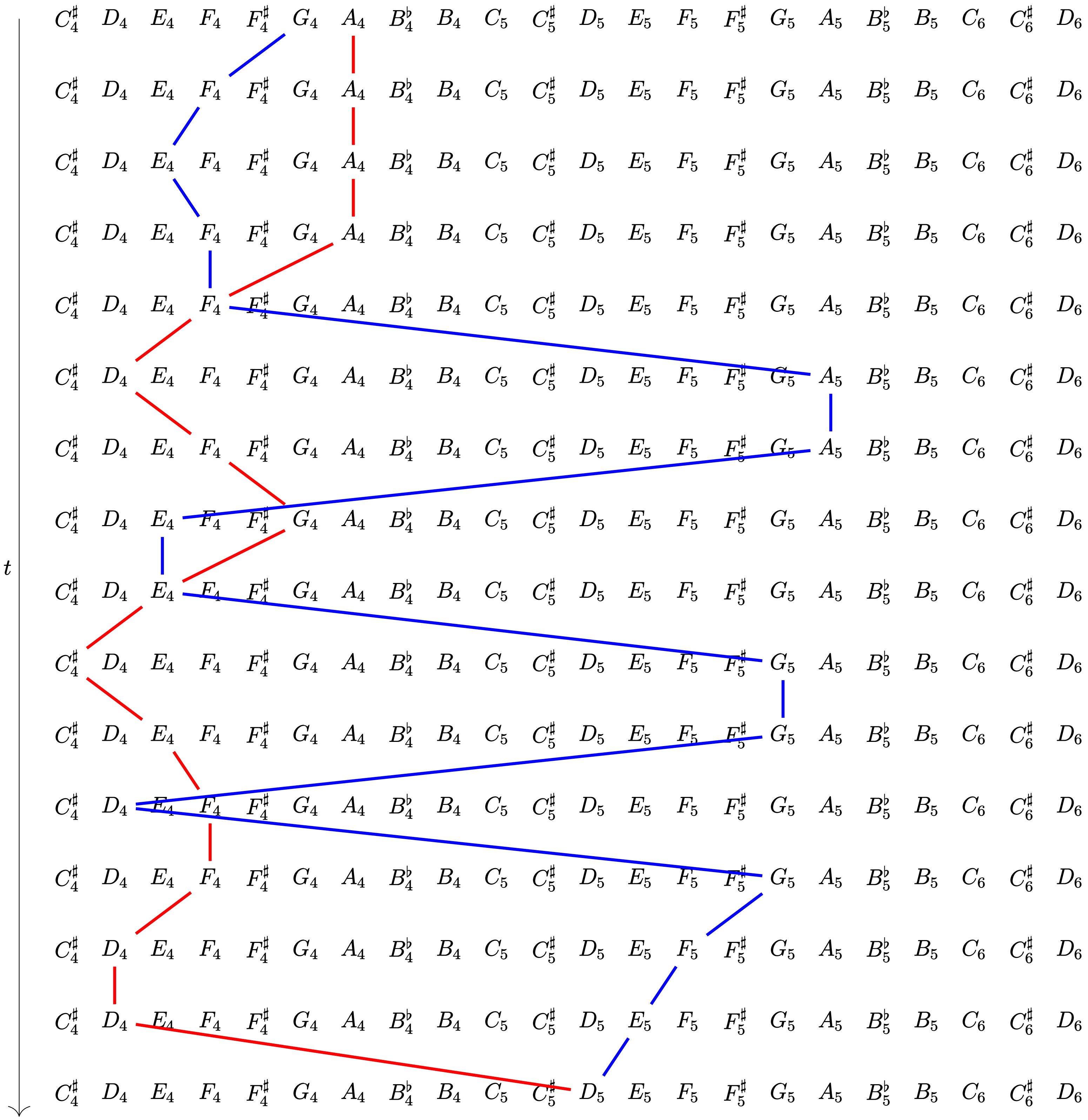
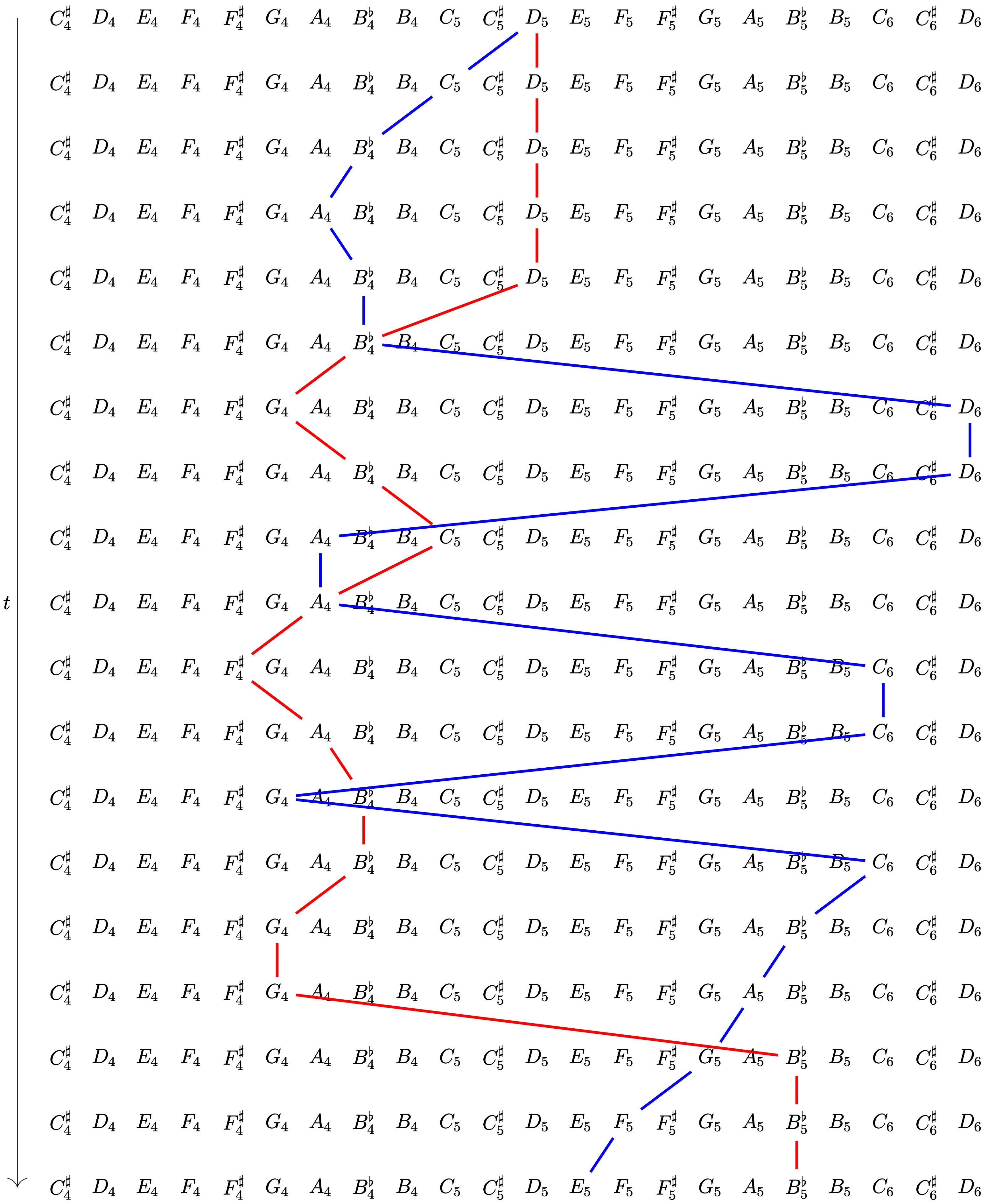
A | Appendix: Voice Leading of Bach's Two Violins

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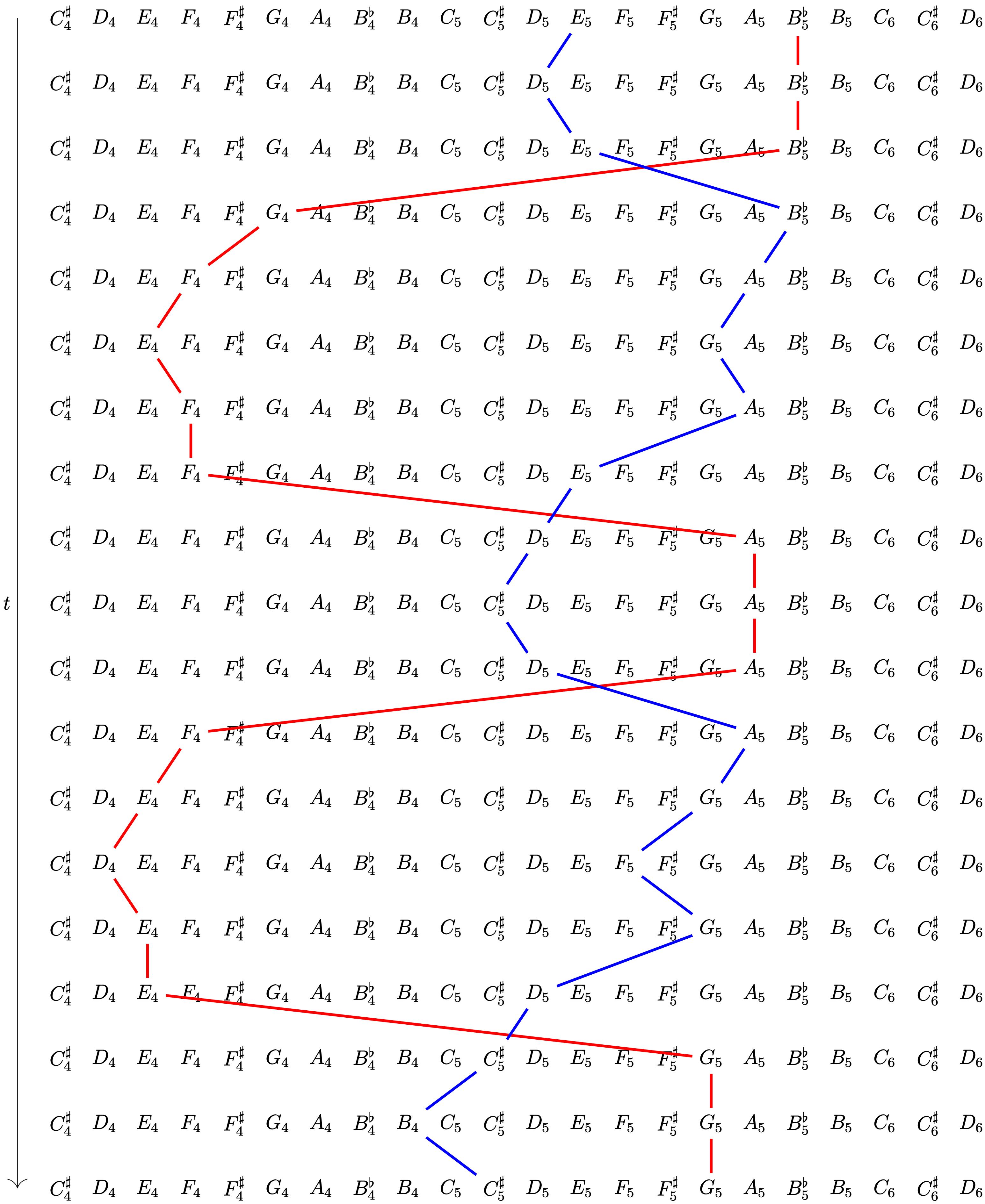
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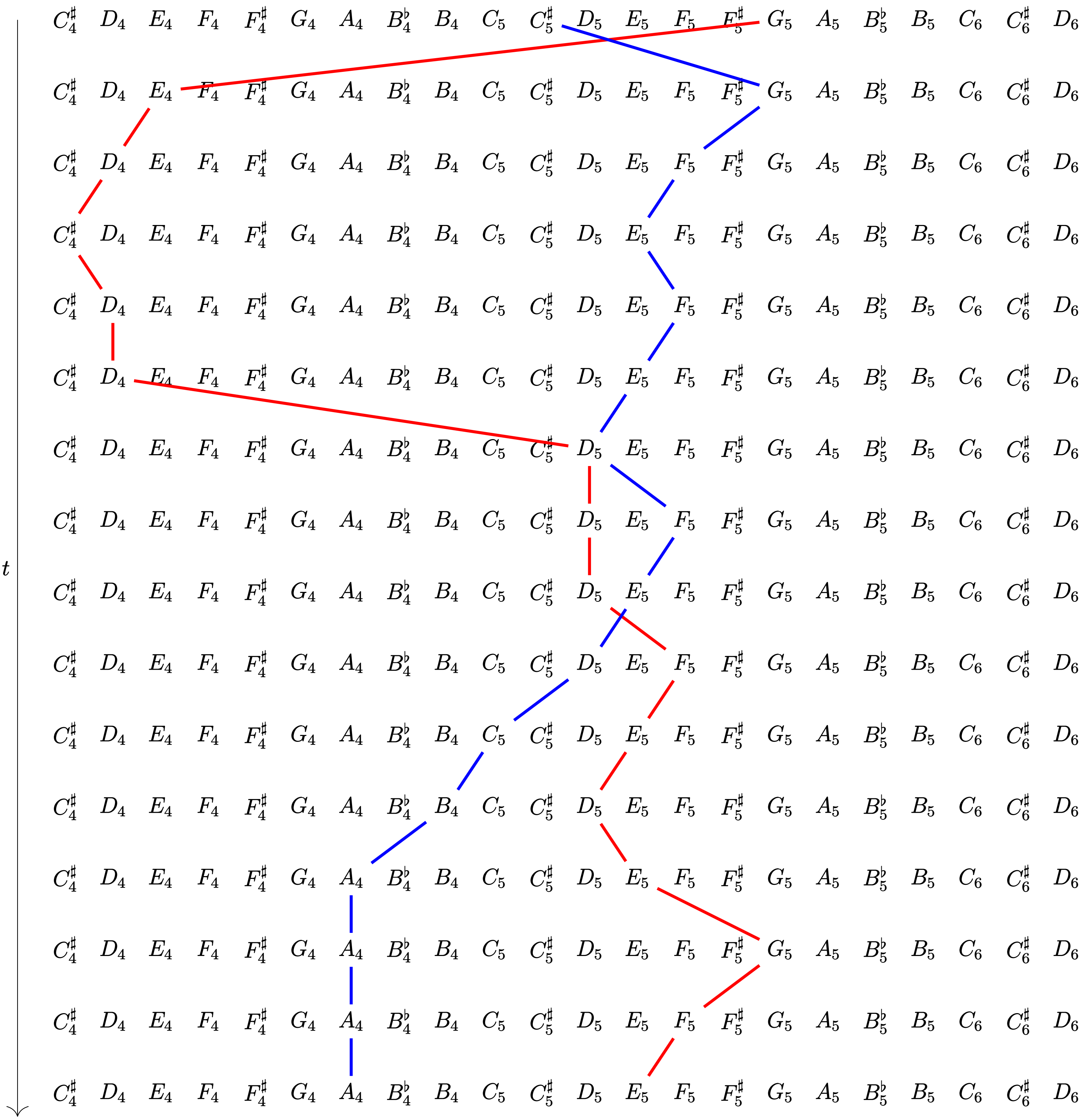
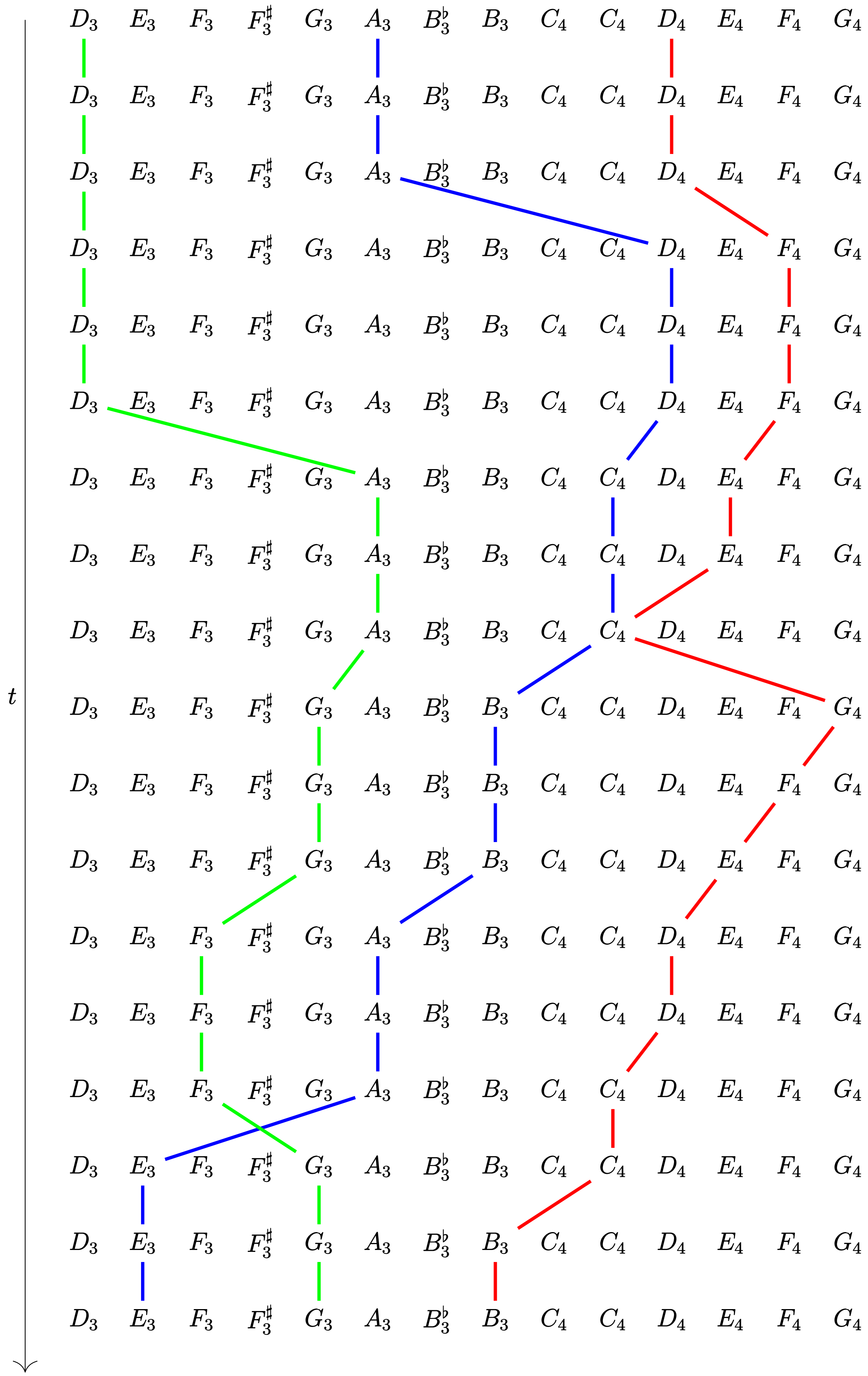


Figure A.1: Voice leading of snippet from Bach Concerto for 2 violins in D minor. Discretised into 16th notes. The first voice is represented by the red line, and the second violin is the blue line.

B | Appendix: Voice Leading of Dufay's Mon Chier Amy



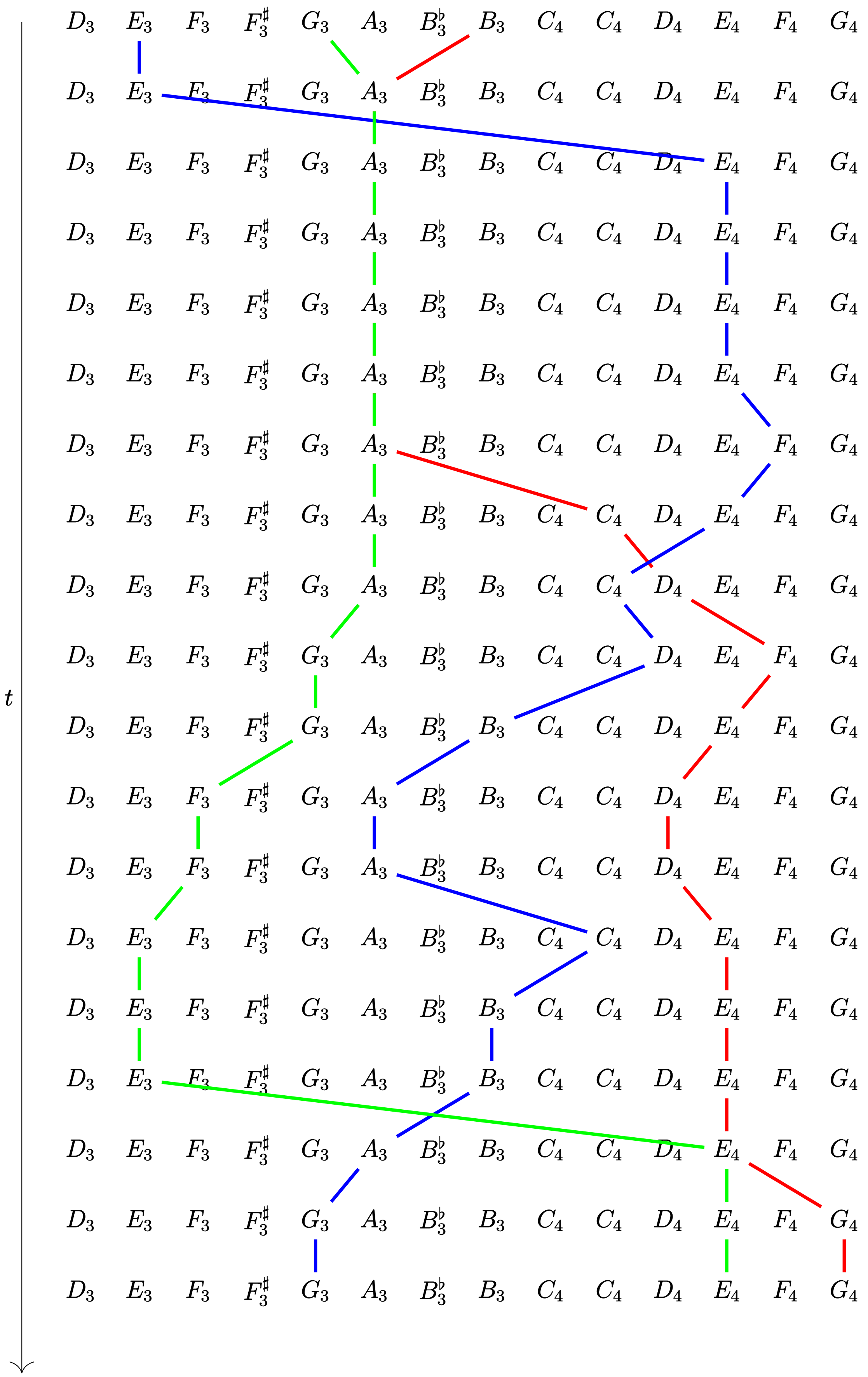


Figure B.1: Voice leading of *Mon chier amy*, discretised into 8th notes. The first voice is represented by the red line, the second voice is the blue line, and the green line is the third voice.

C | Appendix: Genus dependent on Number of Crossings

Proposition C.1. *If a voice leading with two voices contains k voice crossings, the closed surface associated to the Seifert surface from Seifert's algorithm has genus $k-1$.*

Proof. The genus of the closed surface is given in (2.37). Filling the formula in for the genus of the Seifert surface in (2.36), we get

$$g_c = 2 + d - f - m + m - 1 \tag{C.1}$$

with f the number of Seifert circles and d the number of crossings. For an n-braid the number of Seifert circles is n, i.e.$f = n$, or in this case $f = 2$. The number of crossings in the braid is the same as the number of voice crossings by construction, thus $d = k$. We conclude $g_c = 1 + k - 2 = k - 1$. □

Proposition C.2. *If a voice leading with two voices has an even number of crossings, $2k$, the genus of a Seifert surface is*

$$g_s = k - 1. \tag{C.2}$$

If a voice leading with two voices has an uneven number of crossings, $2k+1$, the genus of a Seifert surface is

$$g_s = k. \tag{C.3}$$

Proof. To assess the number of components in a braid, variable m in the formula (2.37), we need to consider the permutation. For a braid with two strands, it is easy to see that an even number of crossings will result in the identity permutation and thus in two components, and an uneven number of crossings will result in one component. □

D | Appendix: Theorem of Mirrored, Reversed, and Flipped Braid

Theorem D.1. *If any braid $\beta \in B_n$ adheres to a partial ordering, so does its mirror braid* $\text{Mir}(\beta)$*, its flipped braids* $\text{Flip}(\beta)$*, and its reverse* $\text{Rev}(\beta)$*.*

Note that the mirror, flipped, and reverse braid do not have to adhere to the same ordering as the initial braid!

Proof. Let use notate the following for a strand i: $o(i)$ is the order of the strand, and $p(i)$ is the position of strand i.

For the mirror braid, the argument is quite easy. Since only the types of crossings are switched, the order relation also switches. That is, for strands s_1 and s_2 such that $p(s_1) = i$ and $p(s_2) = i + 1$, we have that an arbitrary generator $\sigma_i^{\pm 1}$ induces either $o(s_1) > o(s_2)$ or $o(s_1) < o(s_2)$. Flipping the exponent of the generator, we have that $o(s_1) > o(s_2)$ becomes $o(s_1) < o(s_2)$, and vice versa.

We know that the dual ordering of a strict partial ordering R, such that xRy becomes yRx, is also a strict partial ordering.

For the flipped braid, we can refer again to Figure 5.5 for an idea of the process. The flipped braid still represents the same braid, but just the indexing of the strand has changed. This, by the map i_{Flip} from (5.28). Note that Figure 5.5 suggests that the partial ordering is the same but with flipped relations, i.e., the dual ordering, as with the mirror braid, up to that change in indexing. We shall prove by induction that if the jth crossing of the flipped braid defines the dual relation of the jth crossing of the initial braid, then this also works for the $(j+1)$th crossing. Making the flipped braid abide to a valid (dual) partial ordering.

Let us consider the first flipped crossing $\sigma_{n-i_1}^{\pm 1}$ from the initial braid $\beta = \sigma_{i_1}^{e_1} \dots \sigma_{i_k}^{e_k}$[22]. The element $\sigma_{n-i_1}^{\pm 1}$ defines a partial ordering on strands $n - i_1$ and $n - i_1 + 1$. By mapping i_{Flip}, we know that this implies a partial ordering of initial strands $i_1 + 1$ and i_1 respectively. Since the exponent did not change, if we have in the initial braid $o(i) < o(i+1)$ for a positive exponent (respectively $o(i) > o(i+1)$ for a negative exponent), we get $o(i+1) < o(i)$ (respectively $o(i+1) > o(i)$) in the flipped braid.

Without loss of generality, let us assume that the coming crossings are positive. Also, note that the Flip map from Definition 5.20, assumes an endomorphism Flip_{S_n} for the group S_n, by following scheme:

$$\begin{array}{ccc} B_n & \xrightarrow{M} & S_n \\ {\scriptstyle \text{Flip}}\downarrow & & \downarrow{\scriptstyle \text{Flip}_{S_n}} \\ B_n & \xrightarrow{M} & S_n \end{array}$$

with M from Definition 2.42. The map Flip_{S_n} is then defined by its generators as follows:

$$\text{Flip}_{S_n} : S_n \to S_n : (i \;\; i+1) \mapsto (n-i \;\; n-i+1), \qquad \forall 1 \leq i \leq n-1, \tag{D.1}$$

Or equivalently (as proposed from [8, Th 5.11])

$$\text{Flip}_{S_n} : S_n \to S_n : \pi \mapsto M(\Delta_n)\pi M(\Delta_n). \tag{D.2}$$

Recall that $M(\Delta_n^{-1}) = M(\Delta_n)$. This is now clearly also an automorphism. With this map, we can track the permutations of the flipped braid via the permutations of the initial braid.

Let us assume that all previous crossings from the initial and flipped braid were dual to each other, up to indexing $M(\Delta_n)$. If we have a jth crossing σ_{i_j} and two strands s_1 and s_2 such that $p(s_1) = i_j$ and $p(s_2) = i_j + 1$, we will get a relation $o(s_1) < o(s_2)$. The permutation of the previous crossings induced by M, up to j, is the permutation $\pi_j \in S_n$ (which is the inverse of p), with $\pi_j(i_j) = s_1$ and $\pi_j(i_j + 1) = s_2$. We define this in order to mark which strands we have applied a relation on to, i.e., π_j tells us that we have an order on s_1 and s_2. If the flipped braid induces the dual on the same strands, we have proven the requested.

To avoid confusion in indexing for the flipped and initial braid, we use $[.]_{1,2}$ to denote if we talk about a strand index for the initial or flipped braid respectively, e.g., $\pi_j(i_j) = [s_1]_1$. In the flipped braid, we then get the permutation $\text{Flip}_{S_n}(\pi_j) = M(\text{Flip}(\beta_j)) =: \pi_j^* \in S_n$, with β_j the braid up to the jth crossing. And thus, the strand in the flipped braid at position $n - i_j$ is $[\pi_j^*(n - i_j)]_2$, which is the following:

$$\pi_j^*(n - i_j) = \Big[(M(\Delta_n)\pi_j M(\Delta_n))(n - i_j)\Big]_2, \tag{D.3a}$$

$$= \Big[M(\Delta_n)\Big(\pi_j(i_j + 1)\Big)\Big]_2, \tag{D.3b}$$

$$= [M(\Delta_n)(s_2)]_2. \tag{D.3c}$$

[22]The element $\sigma_{n-i_1}^{\pm 1}$ will always be the first in the braid word, even with $|e_1| > 1$.

But note that this indexing, is the indexing for the flipped braid. Thus to get the strand index from the initial braid we apply the inverse of i_{Flip} which is again just the permutation $M(\Delta_n)$. and thus we get that $\pi_j^*(n - i_j) = [s_2]_1$, meaning that in the flipped braid, the crossing σ_{n-i_j} acts on the strand s_2 in a dual way w.r.t. the crossing σ_{i_j} in the initial braid. Similarly, we get that $\pi_j^*(n - i_j + 1) = [s_1]_1$. Thus, a flipped crossing σ_{n-i_j} induces the dual ordering of the initial crossing. Hence, by induction we have that the flipped braid adheres to a partial ordering.

As for the reverse of $\beta \in B_n$, we can do something very similar, but now the map between S_n and S_n is the map Rev_{S_n}, such that we have:

$$\begin{array}{ccc} B_n & \xrightarrow{M} & S_n \\ {\scriptstyle\text{Rev}}\downarrow & & \downarrow{\scriptstyle\text{Rev}_{S_n}} \\ B_n & \xrightarrow{M} & S_n \end{array}$$

with

$$\text{Rev}_{S_n} : S_n \to S_n : (i_1 \;\; i_1+1)^{e_1} \dots (i_k \;\; i_k+1)^{e_k} \mapsto (i_k \;\; i_k+1)^{e_k} \dots (i_1 \;\; i_1+1)^{e_1} \tag{D.4}$$

Notice that because the generators are 2-cycles, this map Rev_{S_n} is basically the inverse map $[\,.\,]^{-1} : S_n \to S_n : \pi \mapsto \pi^{-1}$. We thus have the following:

$$M(\beta)M(\text{Rev}(\beta)) = M(\beta)\text{Rev}_{S_n}(M(\beta)) = 1. \tag{D.5}$$

Thus, w.r.t the initial braid, our permutation of the reversed braid unfolds itself to 1.

Let us consider a braid β of length m, i.e., β has m crossings. If we have thus strands s_1 and s_2 such that $p(s_1) = i_j$ and $p(s_2) = i_j + 1$ for a jth crossing σ_{i_j} in the initial braid, we have $o(s_1) < o(s_2)$. Again there is a permutation $\pi_j \in S_n$ induced by M from before the crossing j in the initial braid such that $\pi_j(i_j) = [s_1]_1$ and $\pi_j(i_j + 1) = [s_2]_1$, using the same $[.]_{1,2}$ notation as before. As with the flipped braid, we can construct an index map as follows for a specific braid β:

$$i_\beta : \{1, \dots, n\} \to \{1, \dots, n\} : i \mapsto (M(\beta))^{-1}(i) \tag{D.6}$$

There exists a permutation that denotes the future permutations after j, $\pi_{m-j-1} \in S_n$, such that

$$M(\beta) = \pi_j \circ (i_j \; i_j + 1) \circ \pi_{m-j-1}. \tag{D.7}$$

By above reasoning we have that the permutation $\text{Rev}_{S_n}(\pi_{m-j-1})$ is the permutation coming from the reversed braid such that it is followed by that same jth crossing, but the relation is switched (over- and undercrossing). We have the following from (D.7).

$$\text{Rev}_{S_n}(\pi_{m-j-1}) = M(\text{Rev}(\beta)) \circ \pi_j \circ (i_j \; i_j + 1), \tag{D.8}$$

with $\pi_j^{-1} = \text{Rev}(\pi_j)$ and thus $\text{Rev}^2(\pi_j) = \pi_j$. Thus, at the $(m - j)$th crossing of the reversed braid, which would be the same as the jth crossing in the initial braid, we should have the same strands s_1 and s_2. We have the following:

$$\text{Rev}_{S_n}(\pi_{m-j-1})(i_j) = [[M(\text{Rev}(\beta)) \circ \pi_j \circ (i_j \; i_j + 1)](i_j)]_2 \tag{D.9a}$$

$$= \Big[[M(\text{Rev}(\beta)) \circ \pi_j](i_j + 1)\Big]_2, \tag{D.9b}$$

$$= \Big[[M(\text{Rev}(\beta))](s_2)\Big]_2. \tag{D.9c}$$

From (D.6), we see that the inverse of this index map $M(\beta)$ is. Recall from (D.6) that we have $M(\mathrm{Rev}(\beta)) = M(\beta)^{-1}$. Thus, we will get that $\mathrm{Rev}_{S_n}(\pi_{m-j-1})(i_j) = [s_2]_1$. Similarly, we have that $\mathrm{Rev}_{S_n}(\pi_{n-j-1})(i_j + 1) = [s_1]_1$ and thus the relation is again turned around for all crossings, resulting in the dual ordering.

□

Corollary D.2. *If a braid β respects a partial ordering of strand, then also the inverse β^{-1}.*

Proof. Note that for a braid β we have the following

$$\beta^{-1} = \mathrm{Mir}(\mathrm{Rev}(\beta)). \tag{D.10}$$

□